Master 1 internship report
Dyadic Harmonic Analysis

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Introduction

First, I would like to thank my mentors who endured me and my ignorance. I learnt many things with them.

Un petit mot en français pour remercier ma famille et mes amis (de l'ENS et d'ailleurs) : c'est grâce à leur soutien que j'ai réussi à devenir normalien cette année.

The purpose of my internship was to become familiar with the dyadic analysis, especially dyadic harmonic analysis. I learnt a lot of things in modern Fourier analysis which theory has only fifty years of existence. Among the most famous people who developped this theory we can quote Calderon, Zygmund, Littlewood, Paley, Meyer, Tao and so on...

The first part of this internship report is here to set all the definitions and small results we will need in the last two parts.

In the second part, we give the proof of the boundedness of the square operator in different cases, looking at its linearization. This is in this part that I had to make, thanks to advises from my mentors, my own proofs of certain results already known, using their new methods.

In the third and last part, we present a dyadic proof of the $T(1)$-theorem which gives a necessary and sufficiant condition for the L^2-boundedness of singular integral operators. The proof is based on Haar functions.

Notations

We write :
a.e. to say "almost every(where)".
iff. to say "if and only if".
$n \in \mathbb{N}^*$ fixed.
(X, μ) a measure space (the measure is always supposed σ-finite).
$\mathcal{B}(\mathbb{R}^n)$ the Borel algebra on $\mathbb{R}^n$.
L^p the Lebesgue space of order $p \in [1, \infty]$.
p' the conjugate exponent of p: $\frac{1}{p} + \frac{1}{p'} = 1$.
$L^1_{loc}(\mathbb{R})$ the set of all locally integrable measurable functions on $\mathbb{R}$.
$\mathcal{S}(\mathbb{R}^n)$ the Schwartz space.
l^p the space of sommable sequences at order $p \in [1, \infty]$.
$\mathbb{E}[.|.]$ the operator of conditional expectation.
$\Delta = \{(x, x)/x \in \mathbb{R}^n\}$ the diagonal of $\mathbb{R}^n$.
$\mathcal{Q}$ the set of cubes in $\mathbb{R}^n$ (i.e. balls for the uniform metric).
$A \lesssim B$ to say $\exists C \in \mathbb{R}, A \leqslant CB$.
$A \approx B$ to say $\exists C \in \mathbb{R}, C^{-1}B \leqslant A \leqslant CB$.
λI the interval with the same center than the interval I but λ larger (for $\lambda \in \mathbb{N}$).

1 Necessary definitions and results to begin

1.1 Dyadic analysis on the line

1.1.1 Dyadic intervals

Definition 1. *We call dyadic interval an interval of the form* $I_{k,n} = [k2^n, (k+1)2^n[$, *where* $(k,n) \in \mathbb{Z}$. n *(or sometimes* 2^n*) is called the scale of the dyadic interval* $I_{k,n}$.
We write $\mathcal{D}$ *the set of all the dyadic intervals of* $\mathbb{R}$.
We write $\mathcal{D}^1$ *the set of all the dyadic intervals contained in* $[0,1]$.

We have the immediate properties :

Proposition 1.

- $\forall n \in \mathbb{Z}, \forall x \in \mathbb{R}, \exists! I_{k,n}, x \in I_{k,n}$.
- $\forall I = [a,b[\in \mathcal{D}, I_l := [a, \frac{a+b}{2}[\in \mathcal{D}$ *and* $I_r := [\frac{a+b}{2}, b[\in \mathcal{D}$.
- $\forall (I,J) \in \mathcal{D}^2$, *if* $I \cap J \neq \emptyset$, *then* $I \subset J$ *or* $J \subset I$.

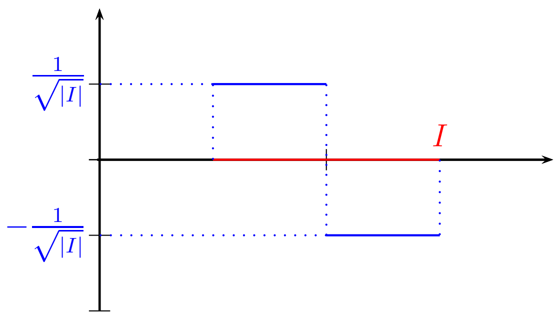
Remark : I_l is called the left-son of I and I_r is called the right-son of I. I is called the parent of I_l and I_r. We write $\widetilde{I_l} = I$ and $\widetilde{I_r} = I$.
$\mathcal{D}^1$ can be represented as a binary tree, where the root is $[0,1]$, and the above vocabulary comes from the informatical theory of trees.

1.1.2 Haar functions

Definition 2. *For all* $I \in \mathcal{D}$, *we set*

$$h_I = \frac{1}{\sqrt{|I|}}(1_{I_l} - 1_{I_r}).$$

h_I *is called the Haar function on the dyadic interval* I.



Remark : Haar functions are the more simple example of wavelets (revolutionary theory in Fourier analysis developped by Meyer).

Definition 3. *Let* $(E, \|.\|)$ *be a normed vector space on a field* K. *Let* $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$.
We say that $(e_n)_{n\in\mathbb{N}}$ *is a Schauder basis in* E *if*

$$\forall x \in E, \exists! (x_n) \in K^{\mathbb{N}}, \lim_{n\to+\infty} \|x - \sum_{k=0}^{n} x_k e_k\| = 0$$

and in this case, we write

$$x = \sum_{n=0}^{\infty} x_n e_n.$$

Remarks :

- Thanks to the property of uniqueness, the family $(e_n)_{n\in\mathbb{N}}$ is free.
- A Hilbert basis is a special case of Schauder basis in a Hibert space.

Lemma 1. *For all $p \in [1,\infty[$, $(h_I)_{I\in\mathcal{D}^1}$ is a Schauder basis in $L^p([0,1])$, orthonormal in $L^2([0,1])$.*

Proof : We just prove that it is a Hilbert basis of $L^2([0,1])$, because it is only that fact we will use in the following parts.

- Let $(I,J) \in (\mathcal{D}^1)^2$.
  $*$ If $I\cap J = \emptyset$, then obviously, $\displaystyle\int_0^1 h_I(x)h_J(x)dx = 0$.
  $*$ If $I = J$, then $\displaystyle\int_0^1 h_I(x)h_J(x)dx = \int_0^1 (h_I(x))^2dx = \frac{1}{|I|}\int_0^1 1_I(x)dx = 1$.
  $*$ If $I\cap J \neq \emptyset$ and $I \neq J$, then $I \subsetneq J$ or $J \subsetneq I$, by properties of dyadic intervals. By symmetry of the roles of I and J, let's assume that $I \subsetneq J$. Then $I \subset J_l$ or $I \subset J_r$. Still by symmetry and properties of dyadic intervals, let's assume that $I \subset J_l$. Then

$$\int_0^1 h_I(x)h_J(x)dx = \frac{1}{(|I||J|)^{\frac{1}{2}}}\int_0^1 1_{I_l}(x) - 1_{I_r}(x)dx = \frac{1}{(|I||J|)^{\frac{1}{2}}}(\frac{|I|}{2} - \frac{|I|}{2}) = 0.$$

- Let $f \in \overline{Vect((h_I)_{I\in\mathcal{D}^1})}^{\perp}$. Then $\forall I \in \mathcal{D}^1$, $\displaystyle\int_0^1 f(x)h_I(x)dx = 0$.
  $*$ We consider $F : \begin{array}{ccc} [0,1] & \mapsto & \mathbb{C} \\ y & \to & \displaystyle\int_0^y f(x)dx \end{array}$.
  Let $I \in \mathcal{D}^1$. We can write

$$h_I = h^m = 2^{\frac{n}{2}}(1_{[\frac{k}{2^n},\frac{k}{2^n}+\frac{1}{2^{n+1}}[} - 1_{[\frac{k+1}{2^n}-\frac{1}{2^{n+1}},\frac{k+1}{2^n}[})$$

  where (n,k) is such that $m = 2^n + k$ and $k \in [\![0, 2^n-1]\!]$.
  Then, the orthogonal hypothesis imposes

$$\int_{\frac{2k}{2^{n+1}}}^{\frac{2k+1}{2^{n+1}}} 2^{\frac{n}{2}}f(x)dx - \int_{\frac{2k+1}{2^{n+1}}}^{\frac{2k+2}{2^{n+1}}} 2^{\frac{n}{2}}f(x)dx = 0$$

  which gives

$$-F(\frac{2k}{2^{n+1}}) + 2F(\frac{2k+1}{2^{n+1}}) - F(\frac{2k+2}{2^{n+1}}) = 0.$$

  Thanks to a trivial induction we have that $F(d) = 0$ for all dyadic number in $[0,1]$.
  $*$ Moreover, thanks to the Cauchy-Schwarz inequality, we have

$$\forall (x,y)\in[0,1]^2, |F(x)-F(y)| = |\int_0^1 f(t)1_{[x,y]}(t)dt| \leqslant \|f\|_2\sqrt{|x-y|}.$$

  Hence, F is continuous on $[0,1]$ and equal to 0 on the dense subset of $[0,1]$ formed by the dyadic numbers. So $F = 0$ and then $f = 0$.
  Thus, $\overline{Vect((h_I)_{I\in\mathcal{D}^1})}^{\perp} = \{0\}$, and then $(h_I)_{I\in\mathcal{D}^1}$ is a Hilbert basis of $L^2([0,1])$.

Remark : In the previous proof, we can see that the Haar system $(h_I)_{I\in\mathcal{D}}$ still forms a Hilbert basis of $L^2(\mathbb{R})$.

1.2 Some important spaces

In this paragraph, we introduce spaces we will work with. We present them in a general background, but we will only use them in $\mathbb{R}$. The spaces presented here are often used in PDE and in Fourier analysis.

1.2.1 Weak-L^p spaces

This is an extansion of L^p spaces. We use them to deduce, through the Marcinkiewicz interpolation theorem (Theorem 1), the L^p boundedness of some operators.

Definition 4. *Let $p \in [1, \infty[$. Let (X, μ) be a mesure space. We call weak-L^p space on (X, μ) the set of $\mu-$mesurable functions such that*

$$\sup_{\lambda>0} \lambda^p \mu\{x \in X / |f(x)| > \lambda\} < \infty.$$

We write it $L^{p,\infty}(X, \mu)$.
Then we define the following semi-norm on this space by

$$\|f\|_{p,\infty} = \left(\sup_{\lambda>0} \lambda^p \mu\{x \in X / |f(x)| > \lambda\}\right)^{\frac{1}{p}}.$$

We set $L^{\infty,\infty}(X, \mu) = L^\infty(X, \mu)$.

Remarks : Thanks to (Tchebychev-)Markov's inequality :

$$\forall \lambda > 0, \lambda^p |\{x \in \mathbb{R}^n / |f(x)| > \lambda\}| \leqslant \|f\|_p^p,$$

we have

$$L^p(X, \mu) \subset L^{p,\infty}(X, \mu).$$

But we don't have the equality. For example in $\mathbb{R}$, we consider $f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array}$. We know that $f \notin L^1(\mathbb{R})$ (Riemann integral), but $\forall \lambda > 0, |\{x \in \mathbb{R} / |f(x)| > \lambda\}| = |] - \frac{1}{\lambda}, \frac{1}{\lambda}[\backslash\{0\}| = \frac{2}{\lambda}$.
So

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R} / |f(x)| > \lambda\}| = 2 < +\infty.$$

So $f \in L^{1,\infty}(\mathbb{R})$.

1.2.2 Hardy spaces

We do not define Hardy spaces as it is usual to do, but by an equivalant : the atomic decomposition.
Moreover, we consider the dyadic form of this space (because we will need dyadic properties in section two).
We write $\mathcal{Q}_d$ the set of all dyadic cubes in $\mathbb{R}^n$ (i.e. cubes with dyadic sides).

Definition 5. *Let $p \in [1, \infty]$. Let $Q \in \mathcal{Q}_d$. Let $a : Q \to \mathbb{C}$ be a mesurable function. We say that a is a p-atom on Q if it satisfies :*

- *$supp(a) \subset Q$.*
- $\|a\|_p \leqslant \frac{1}{|Q|^{1-\frac{1}{p}}}$.

- $\displaystyle\int_Q a(x)dx = 0.$

We denote the collection of p-atoms on Q by $\mathcal{A}_Q^p$, and we set $\mathcal{A}^p = \bigcup_{Q\in\mathcal{Q}_d} \mathcal{A}_Q^p$.

Definition 6. *Let $p \in]1,\infty]$. We define the p-Hardy space as*

$$H^{1,p} = \{f \in L^1(\mathbb{R}^n)/\exists (a_i)_{i\in\mathbb{N}} \in (\mathcal{A}^p)^{\mathbb{N}}, \exists(\lambda_i)_{i\in\mathbb{N}} \in l^1(\mathbb{N},\mathbb{C}), f = \sum_{i=0}^{\infty} \lambda_i a_i)\}.$$

We define the norm associated :

$$\|f\|_{H^{1,p}} = \inf\{\|\lambda\|_{l^1(\mathbb{N},\mathbb{C})}/f = \sum_{i=0}^{\infty} \lambda_i a_i\}.$$

Actually, $(H^{1,p}, \|.\|_{H^{1,p}})$ is a Banach space, and $\forall p \in]1,\infty[$, $H^{1,p} = H^{1,\infty} \subset L^1(\mathbb{R}^n)$, thus we can define the Hardy space as $H^1 = H^{1,p}$ for $p \in]1,\infty]$, associated with its norm (see [AUS12]).
The ideas of the proof :
The inclusion in L^1 is immediate by definition.
We want to show that $(H^{1,p}, \|.\|_{H^{1,p}}) = (H^{1,\infty}, \|.\|_{H^{1,\infty}})$.

- The inclusion $\supseteq$ is easy by Hölder inequality.
- We show $\subseteq$ using a good Calderon-Zygmund decomposition : for every p-atom a, we write $a = b + g$ with $\|b\|_{H^{1,p}} \leqslant \frac{1}{2}$ and $\|g\|_{H^{1,\infty}} \lesssim 1$. Then as every function in $H^{1,p}$ is a sum of p-atoms, we can conclude with an iteration argument.

1.2.3 BMO spaces

Singular integral operators, we introduce in the next paragraph, don't map $L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ (see Theorem 9). The good space to consider is the Bounded Mean Oscillation (BMO) space. This is the space of functions which don't grow too far away from their average on every cube of $\mathbb{R}^n$.

Definition 7. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. The mean oscillation of f in a cube $Q \in \mathcal{Q}_d$ is the number :*

$$⨍_Q |f(x) - f_Q|dx = \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx$$

where $f_Q = \frac{1}{|Q|}\displaystyle\int_Q f(x)dx$.
We set

$$f^{\#}(x) = \sup\{⨍_Q |f(y) - f_Q|dy/x \in Q \in \mathcal{Q}_d\}.$$

Then we say that f has bounded mean oscillation, and we write $f \in BMO(\mathbb{R}^n)$, iff $f^{\#} \in L^\infty(\mathbb{R}^n)$ and we take :

$$\|f\|_{BMO} = \|f^{\#}\|_\infty.$$

Remark : Clearly, $\|.\|_{BMO}$ is a semi-norm and $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, with $\|f\|_{BMO} \leqslant 2\|f\|_\infty$. But we don't have the equality.
Counter example :

Let $f(x) = \ln|x|$. We will show that $f \in BMO(\mathbb{R}) \backslash L^\infty(\mathbb{R})$.
Let $t > 0$. We set

$$f_t(x) = \ln|\frac{x}{t}| = \ln|x| - \ln|t| = f(x) + c.$$

For I an interval of length t, and J an interval of unit length, we have by change of variables :

$$\frac{1}{|I|}\int_I |f(x) - \frac{1}{|I|}\int_I f(y)dy|dx = \frac{1}{|I|}\int_I |f_t(x) - \frac{1}{|I|}\int_I f_t(y)dy|dx = \frac{1}{|J|}\int_J |f(x) - \frac{1}{|J|}\int_J f(y)dy|dx.$$

So we can consider only the case $|I| = 1$.
Let $I = [x_0 - \frac{1}{2}, x_0 + \frac{1}{2}]$ where $x_0 \in \mathbb{R}$.
By symmetry, we can suppose that $x_0 \geqslant 0$.
$*$ If $x_0 \in [0, 3]$, then we set $C_I = 0$ and we have :

$$\frac{1}{|I|}\int_I |f(x)|dx = \int_I |f(x)|dx \leqslant \int_{-\frac{1}{2}}^{\frac{7}{2}} |\ln|x||dx < +\infty.$$

$*$ Otherwise, $x_0 \geqslant 3$.

$$\begin{aligned}
\forall x \in I, |\ln(x) - \ln(x_0 - \tfrac{1}{2})| &= \ln(x) - \ln(x_0 - \tfrac{1}{2}) \\
&= \int_{x_0 - \frac{1}{2}}^{x} \frac{1}{t} dt \\
&\leqslant \int_{x_0 - \frac{1}{2}}^{x} \frac{2}{5} \\
&= \tfrac{2}{5}\left(x - x_0 + \tfrac{1}{2}\right) \\
&\leqslant \tfrac{2}{5}\left(x_0 + \tfrac{1}{2} - x_0 + \tfrac{1}{2}\right) \\
&= \tfrac{2}{5}
\end{aligned}$$

We set $C_I = \ln(x_0 - \frac{1}{2})$, then

$$\frac{1}{|I|}\int_I |\ln(x) - C_I|dx = \int_I |\ln(x) - C_I|dx \leqslant \frac{2}{5}.$$

In every case, we have :

$$\frac{1}{|I|}\int_I |f(x) - C_I|dx \leqslant \max\left(\frac{2}{5}, \int_{-\frac{1}{2}}^{\frac{7}{2}} |\ln|x||dx\right) := C.$$

But

$$f - \frac{1}{|I|}\int_I f(y)dy = f - C_I + C_I - \frac{1}{|I|}\int_I f(y)dy = f - C_I + \frac{1}{|I|}\int_I (C_I - f(y))dy.$$

Hence

$$\frac{1}{|I|}\int_I |f(x) - \frac{1}{|I|}\int_I f(y)dy|dx \leqslant \frac{1}{|I|}\int_I |f(x) - C_I|dx + \frac{1}{|I|}\int_I |C_I - f(y)|dy = 2\frac{1}{|I|}\int_I |f(x) - C_I|dx \leqslant 2C.$$

So $f \in BMO(\mathbb{R})$ and it is obvious that $f \notin L^\infty(\mathbb{R})$.
Thus

$$L^\infty(\mathbb{R}) \subsetneq BMO(\mathbb{R}).$$

An other important result is that we have $(H^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$ (see [AUS12]).
The ideas of the proof :

- We show that $BMO = BMO_p$ where $BMO_p = \{f \in L^p_{loc} / \sup_{Q\in\mathcal{Q}_d} \frac{1}{|Q|}\int_Q |f(x) - \frac{1}{|Q|}\int_Q f(y)dy|dx\}$.
  This is proved using John-Nirenberg's inequality which states that the worst behaviour for a BMO function is to blow up logarithmicaly.
- Then we build the isomorphism between $(H^{1,2})^*$ and BMO_2. The difficulty is to show the surjectivity. Actually, we build the application on atoms and we get the general case by density using the Riesz representation theorem.

1.3 Main results

1.3.1 Interpolation

The idea of the interpolation is to get informations on "intermediate" operators or spaces having informations only on "extremal" ones.

Definition 8. *Let T be an operator on a vector space V of complex-valued measurable functions on (X,μ) and taking values in the set of all complex-valued finite a.e. measurable functions on (Y,ν).*
We say that T is sublinear if

$$\forall (f,g) \in V^2, \forall \lambda \in \mathbb{C}, |T(f+g)| \leqslant |T(f)| + |T(g)| \text{ and } |T(\lambda f)| = |\lambda||T(f)|.$$

Definition 9. *Let T be a sublinear operator. Let $(p,q) \in [1,\infty]^2$. We say that*

- *T is of strong-type (p,q) if $T : L^p \to L^q$ is bounded, i.e.*

$$\exists C > 0, \forall f \in L^p, \|Tf\|_q \leqslant C\|f\|_p.$$

- *T is of weak-type (p,q) if $T : L^p \to L^{q,\infty}$ is bounded, i.e.*

$$\exists C > 0, \forall f \in L^p, \|Tf\|_{q,\infty} \leqslant C\|f\|_p.$$

We give a simple version of the Marcinkiewicz interpolation theorem, the one we will need later. This is the theorem of real interpolation.

Theorem 1 (Marcinkiewicz interpolation theorem)**.** *Let $(p_0,q_0) \in [1,\infty]^2$ such that $p_0 < q_0$. Let T be a sublinear operateur defined on $L^{p_0}(X,\mu) + L^{q_0}(X,\mu)$ which is of weak-type (p_0,p_0) and weak-type (q_0,q_0).*
Then T is of strong-type (p,p), for all $p \in]p_0,q_0[$.

Proof :

- First let's show that $\forall p \in]p_0,q_0[, L^p(X,\mu) \subset L^{p_0}(X,\mu) + L^{q_0}(X,\mu)$ (to justify that the assertion of the theorem makes sens).
  Let $p \in]p_0,q_0[$. Let $f \in L^p(X,\mu)$. Let $\alpha > 0$.
  We set $f_{1,\alpha} = f1_{|f|>\alpha}$ and $f_{2,\alpha} = f1_{|f|\leqslant\alpha}$. Thus $f = f_{1,\alpha} + f_{2,\alpha}$.
  Moreover,

$$\int_X |f_{1,\alpha}(x)|^{p_0} d\mu(x) = \int_X |f_{1,\alpha}(x)|^p |f_{1,\alpha}(x)|^{\overbrace{p_0 - p}^{<0}} d\mu(x) \leqslant \alpha^{p_0-p}\int_X |f_{1,\alpha}(x)|^p d\mu(x) < +\infty$$

and

$$\int_X |f_{2,\alpha}(x)|^{q_0} d\mu(x) = \int_X |f_{2,\alpha}(x)|^p |f_{2,\alpha}(x)|^{\overbrace{q_0 - p}^{>0}} d\mu(x) \leqslant \alpha^{q_0-p} \int_X |f_{2,\alpha}(x)|^p d\mu(x) < +\infty.$$

So $f_{1,\alpha} \in L^{p_0}(X,\mu)$ and $f_{2,\alpha} \in L^{q_0}(X,\mu)$.
So

$$L^p(X,\mu) \subset L^{p_0}(X,\mu) + L^{q_0}(X,\mu).$$

- Let $f \in L^p(X,\mu)$ fixed until the end of the proof.
  We show that $\|Tf\|_p^p = p\int_0^{+\infty} \alpha^{p-1}\mu\{|Tf| > \alpha\}d\alpha$. Actually this an immediate consequence of the theorem of Fubini-Tonelli (F-T) :

$$\begin{aligned} p\int_0^{+\infty} \alpha^{p-1}\mu\{|Tf| > \alpha\}d\alpha &= \int_0^{+\infty} p\alpha^{p-1}\int_X 1_{|Tf|>\alpha}(x)d\mu(x)d\alpha \\ &= \int_X \int_0^{+\infty} p\alpha^{p-1} 1_{|Tf|>\alpha}(x) d\alpha d\mu(x) \\ &= \int_X |Tf(x)|^p d\mu(x) = \|Tf\|_p^p. \end{aligned}$$

- We decompose f as before : $f = f_{1,\alpha} + f_{2,\alpha}$ for a given $\alpha > 0$.
  We have :

$$\begin{aligned} \mu\{|Tf| > \alpha\} &\underset{T \text{ sublinear}}{\leqslant} \mu\{|Tf_{1,\alpha}| > \tfrac{\alpha}{2}\} + \mu\{|Tf_{2,\alpha}| > \tfrac{\alpha}{2}\} \\ &\underset{T \text{ weak-type}}{\leqslant} C\left(\left(\tfrac{2}{\alpha}\|f_{1,\alpha}\|_{p_0}\right)^{p_0} + \left(\tfrac{2}{\alpha}\|f_{2,\alpha}\|_{q_0}\right)^{q_0}\right). \end{aligned}$$

  where $C = \max(\|T\|_{p_0\to(p_0,\infty)}, \|T\|_{q_0\to(q_0,\infty)})$.
  Hence :

$$\begin{aligned} \|Tf\|_p^p &\leqslant p\int_0^{+\infty} \alpha^{p-1}\mu\{|Tf_{1,\alpha}| > \frac{\alpha}{2}\}d\alpha + p\int_0^{+\infty} \alpha^{p-1}\mu\{|Tf_{2,\alpha}| > \frac{\alpha}{2}\}d\alpha \\ &\leqslant C\left(p\int_0^{+\infty} \alpha^{p-1}\left(\frac{2}{\alpha}\|f_{1,\alpha}\|_{p_0}\right)^{p_0} d\alpha + p\int_0^{+\infty} \alpha^{p-1}\left(\frac{2}{\alpha}\|f_{2,\alpha}\|_{q_0}\right)^{q_0} d\alpha\right) \\ &\underset{\text{F-T}}{\leqslant} C2^{q_0}p\left(\int_0^{+\infty} \alpha^{p-1-p_0}\|f_{1,\alpha}\|_{p_0}^{p_0} d\alpha + \int_0^{+\infty} \alpha^{p-1-q_0}\|f_{2,\alpha}\|_{q_0}^{q_0} d\alpha\right) \\ &\leqslant C2^{q_0}p\left(\int_X \int_0^{+\infty} \left(|f(x)|^{p_0}1_{|f|>\alpha}(x)\alpha^{p-1-p_0} + |f(x)|^{q_0}1_{|f|\leqslant\alpha}(x)\alpha^{p-1-q_0}\right) d\alpha d\mu(x)\right) \\ &\leqslant C2^{q_0}p\left(\tfrac{1}{p-p_0} + \tfrac{1}{q_0-p}\right)\|f\|_p^p. \end{aligned}$$

Remark : We can reduce the hypothesis and replace "weak-type (r,r)" by "for all A measurable subset of X, $\|T(1_A)\|_{r,\infty} \lesssim \mu(A)^{\frac{1}{r}}$."
This is called the restricted type of interpolation (see [GRAF108]).

Now we give the theorem of complex interpolation : Riesz-Thorin interpolation theorem. We speak about "complex" interpolation beacause the proof uses complex analysis.

Theorem 2 (Riesz-Thorin interpolation theorem (see (GRAF108))). *Let $(p_0,q_0) \in [1,\infty]^2$ such that $p_0 < q_0$. Let T be a $\mathbb{C}$-linear operateur defined on $L^{p_0}(X,\mu) + L^{q_0}(X,\mu)$ which is of strong-type (p_0,p_0) with constant C_{p_0} and of strong-type (q_0,q_0) with constant C_{q_0}.*
Then T is of strong-type (p,p), for all $p \in]p_0,q_0[$ with constant $C \leqslant C_{p_0}^{1-\theta}C_{q_0}^{\theta}$ where $\theta \in]0,1[$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{q_0}$.

1.3.2 Hardy-Littlewood maximal operator

Definition 10. *The Hardy-Littlewood maximal function is defined by*

$$\forall x \in \mathbb{R}^n, \forall f \in L^1_{loc}(\mathbb{R}^n), Mf(x) = \sup\{\frac{1}{|Q|}\int_Q |f(y)|dy / x \in Q \in \mathcal{Q}\}.$$

The Hardy-Littlewood operator is defined by

$$M : f \mapsto Mf.$$

⚠We are not working with dyadic cubes here.

Proposition 2.

- *M is of weak-type $(1,1)$.*
- *M is of strong-type (p,p), for all $p \in]1,\infty]$.*

Proof :

This proof is the fist example in this intership report where we use a "stopping time" argument : we build a process where at each step we choose a "good" collection of cubes (here) or intervals (further) on which we control a certain quantity.

- Let $f \in L^1_{loc}(\mathbb{R}^n)$.
  $*$ Let $\lambda > 0$. Let $x \in \mathbb{R}^n$ such that $Mf(x) > \lambda$.
  Then, there exists $Q \in \mathcal{Q}$ such that $x \in Q$ and $\frac{1}{|Q|}\int_Q |f(y)|dy > \lambda$.
  Thus, $\{x \in \mathbb{R}^n / Mf(x) > \lambda\}$ is an open set in $\mathbb{R}^n$.
  $*$ Let K be a compact set in $\mathbb{R}^n$ such that

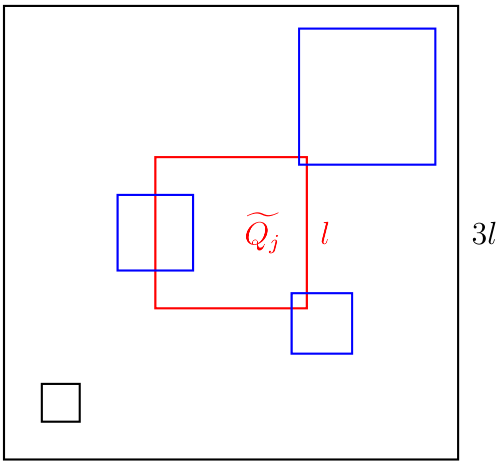
  $$K \subset \{x \in \mathbb{R}^n / Mf(x) > \lambda\}.$$

  With Borel-Lebesgue property, we have $K \subset \bigcup_{i=1}^{N} Q_i$, where $N \in \mathbb{N}^*$ and for all $i \in [\![1,N]\!]$, $Q_i \in \mathcal{Q}$ and $\frac{1}{|Q_i|}\int_{Q_i} |f(y)|dy > \lambda$.
  For that, we give a "stopping-time argument".
  Let $\widetilde{Q_1}$ be such that $\exists i_0 \in [\![1,N]\!], \widetilde{Q_1} = Q_{i_0}$ and $\forall i \in [\![1,N]\!], |Q_i| \leqslant |\widetilde{Q_1}|$
  We keep him and all other Q touching him, we form with them a collection C_1 and we set

  $$I_1 = \{i \in [\![1,N]\!] / Q_i \in C_1\}.$$

  Then, let $\widetilde{Q_2}$ such that $\exists j_0 \in [\![1,N]\!] \backslash I_1, \widetilde{Q_2} = Q_{j_0}$ and $\forall i \in [\![1,N]\!] \backslash I_1, |Q_i| \leqslant |\widetilde{Q_2}|$. And we repeat the process.
  Finally, we have built a family $(\widetilde{Q_j})_{j\in\mathbb{N}}$ such that

  $$|K| \leqslant 3^n \sum_{j\in\mathbb{N}} |\widetilde{Q_j}|.$$



At step j, we get $\widetilde{Q_j}$ (in red) and we keep it and all blue cubes in the collection C_j.
At the end, as the $(\widetilde{Q_j})_{j\in\mathbb{N}}$ are pairewise disjoint, we have :

$$|K| \leqslant 3^n \sum_{j\in\mathbb{N}} |\widetilde{Q_j}| \leqslant \frac{3^n}{\lambda} \sum_{j\in\mathbb{N}} \int_{\widetilde{Q_j}} |f(y)|dy \leqslant \frac{3^n}{\lambda}\|f\|_1.$$

We conclude with the regularity of Lebesgue's measure :

$$|\{x\in\mathbb{R}^n/Mf(x)>\lambda\}| = \sup\{|K|/K\subset\{x\in\mathbb{R}^n/Mf(x)>\lambda\}, K \text{ compact}\} \leqslant \frac{3^n}{\lambda}\|f\|_1.$$

So M is of weak-type $(1,1)$.

- Clearly, $\forall f\in L^\infty(\mathbb{R}^n), \|Mf\|_\infty \leqslant \|f\|_\infty$.
  So, thanks to Marcinkiewicz interpolation theorem, M is of strong-type (p,p) for all $p\in]1,\infty]$.

1.3.3 Square operator

The square operator is the heart of the Littlewood-Paley theory on which my mentors work. Here we present its dyadic (so descreatized) form through Haar functions and its main properties. The next part of this internship report is dedicated to prove those results without using the classical theory, but only "simple" arguments.

Definition 11. *We define the square function associated with some finite collection $\mathcal{I}$ of dyadic intervals by*

$$\forall x\in\mathbb{R}^n, \forall f\in L^1_{loc}(\mathbb{R}), S_{\mathcal{I}}f(x) = \left(\sum_{I\in\mathcal{I}} |\langle f,h_I\rangle|^2|h_I(x)|^2\right)^{\frac{1}{2}}.$$

The square operator is defined by

$$S_{\mathcal{I}} : f \mapsto S_{\mathcal{I}}f.$$

Proposition 3. *The square operator is bounded on $L^2(\mathbb{R})$.*

Proof :
Thanks to the orthogonal property of Haar functions, we have :

$$\|S_{\mathcal{I}}f\|_2 = \left(\int_{\mathbb{R}} \sum_{I\in\mathcal{I}} |\langle f,h_I\rangle h_I(x)|^2 dx\right)^{\frac{1}{2}} = \|\sum_{I\in\mathcal{I}}\langle f,h_I\rangle h_I\|_2 \leqslant \|f\|_2.$$

So the square operator is bounded on $L^2(\mathbb{R})$.

Proposition 4. *The square operator is of weak-type $(1,1)$ and of strong-type (p,p) for all $p \in]1,\infty[$.*

Remark : This is this result we will focus on and prove in the last part using an other method that the usual one based on the Calderon-Zygmund decomposition (see [MS213]).

We will need the linearization of the square operator (which is clearly not linear). For that, we need the following inequalities :

Proposition 5 (Kintchine's inequalities)**.** *Let $p \in [1,\infty[$. Let $m \in \mathbb{N}^*$.*
Then there exists $(A_p, B_p) \in (\mathbb{R}_+^*)^2$ such that for all $(a_n)_{n \in [\![1,m]\!]} \in \mathbb{R}^m$.

$$A_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}} \leqslant \left(\int_0^1 |\sum_{n=1}^{m} a_n r_n(t)|^p dt \right)^{\frac{1}{p}} \leqslant B_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}}.$$

where $\forall n \in [\![1,m]\!], \forall t \in [0,1], r_n(t) = sgn(\sin(2^n \pi t)) \in \{-1,0,1\}$.
The r_n are called the Rademacher functions.

Remark : The result can be written $\left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}} \approx \left(\int_0^1 |\sum_{n=1}^{m} |a_n r_n(t)|^p dt \right)^{\frac{1}{p}}$.

Proof :

- Let $(i,j) \in \mathbb{N}^2$ such that $i \neq j$.

$$\begin{aligned} \int_0^1 r_i(t) r_j(t) dt &= \int_0^1 \sin(2^i \pi t) \sin(2^j \pi t) dt \\ &= \frac{1}{2} \int_0^1 \cos((2^i - 2^j)\pi t) dt - \frac{1}{2} \int_0^1 \cos((2^i + 2^j)\pi t) dt \\ &= \frac{1}{2} \left[\frac{\sin((2^i - 2^j)\pi t)}{(2^i - 2^j)\pi} \right]_0^1 - \frac{1}{2} \left[\frac{\sin((2^i + 2^j)\pi t)}{(2^i + 2^j)\pi} \right]_0^1 \\ &= 0. \end{aligned}$$

  Thanks to this orthogonal property, we have $A_2 = 1 = B_2$.
  Moreover, thanks to the monotonicity of the L^p-norms, we have :

$$\forall (r,p) \in [1,\infty[^2, r \leqslant p \Rightarrow (A_r \leqslant A_p \text{ and } B_r \leqslant B_p).$$

  So we just have to prove that $A_1 > 0$ and $\forall k \in \mathbb{N}, B_{2k} < +\infty$.

- We focus on B_{2k}.
  We set

$$E := \int_0^1 |\sum_{n=1}^{m} a_n r_n(t)|^{2k} dt = \int_0^1 \left(\sum_{n=1}^{m} a_n r_n(t) \right)^{2k} dt.$$

  We have :

$$\begin{aligned} E &\underset{\text{multinomial}}{=} \sum_{|\alpha|=2k} \frac{(2k)!}{\alpha_1! ... \alpha_m!} a_1^{\alpha_1} ... a_m^{\alpha_m} \int_0^1 r_1^{\alpha_1}(t) ... r_m^{\alpha_m}(t) dt \\ &= \sum_{|\alpha|=k} \frac{(2k)!}{(2\alpha_1)! ... (2\alpha_m)!} a_1^{2\alpha_1} ... a_m^{2\alpha_m} \int_0^1 r_1^{2\alpha_1}(t) ... r_m^{2\alpha_m}(t) dt \\ &= \sum_{|\alpha|=k} \frac{(2k)!}{(2\alpha_1)! ... (2\alpha_m)!} a_1^{2\alpha_1} ... a_m^{2\alpha_m} \end{aligned}$$

We have used the fact that $\displaystyle\int_0^1 r_1^{\alpha_1}(t)...r_m^{\alpha_m}(t)dt = \begin{cases} 0 & \text{if } \exists i \in [\![1,m]\!], \alpha_i \equiv 1\ [2] \\ 1 & \text{if } \forall i \in [\![1,m]\!], \alpha_i \equiv 0\ [2] \end{cases}$. But for all $\alpha \in \mathbb{N}^m$ such that $|\alpha| = k$, we have :

$$2^k \alpha_1!...\alpha_m! = (2^{\alpha_1}\alpha_1!)...(2^{\alpha_m}\alpha_m!) \leqslant (2\alpha_1)!...(2\alpha_m)!.$$

Hence,

$$E \leqslant \frac{(2k)!}{2^k k!} \sum_{|\alpha|=k} \frac{k!}{\alpha_1!...\alpha_m!} a_1^{2\alpha_1}...a_m^{2\alpha_m} = \frac{(2k)!}{2^k k!} \left(\sum_{n=1}^m |a_n|^2 \right)^k.$$

Thus,

$$E^{\frac{1}{2k}} \leqslant \left(\frac{(2k)!}{2^k k!} \right)^{\frac{1}{2k}} \left(\sum_{n=1}^m |a_n|^2 \right)^{\frac{1}{2}}.$$

So

$$B_{2k} = \left(\frac{(2k)!}{2^k k!} \right)^{\frac{1}{2k}} < +\infty.$$

- Now we focus on A_1.

  We set $f(t) = \displaystyle\sum_{n=1}^m a_n r_n(t)$.

$$\begin{aligned} \int_0^1 |f(t)|^2 dt &= \int_0^1 |f(t)|^{\frac{2}{3}} |f(t)|^{\frac{4}{3}} dt \\ &\underset{\text{Hölder}}{\leqslant} \left(\int_0^1 |f(t)| dt \right)^{\frac{2}{3}} \left(\int_0^1 |f(t)|^4 \right)^{\frac{1}{3}} \\ &\leqslant \left(\int_0^1 |f(t)| dt \right)^{\frac{2}{3}} B_4^{\frac{4}{3}} \left(\sum_{n=1}^m |a_n|^2 \right)^{\frac{2}{3}} \\ &= \left(\int_0^1 |f(t)| dt \right)^{\frac{2}{3}} B_4^{\frac{4}{3}} \|f\|_2^{\frac{4}{3}}. \end{aligned}$$

Thus,

$$\left(\int_0^1 |f(t)| dt \right)^{\frac{2}{3}} \geqslant B_4^{-\frac{4}{3}} \left(\int_0^1 |f(t)|^2 \right)^{\frac{1}{3}}.$$

Hence

$$\int_0^1 |\sum_{n=1}^m a_n r_n(t)| dt = \int_0^1 |f(t)| dt \geqslant B_4^{-2} \left(\int_0^1 |f(t)|^2 \right)^{\frac{1}{2}} = B_4^{-2} \left(\sum_{n=1}^m |a_n|^2 \right)^{\frac{1}{2}}.$$

So

$$0 < B_4^{-2} \leqslant A_1.$$

Definition 12. *We define the linearization of the operator $S_{\mathcal{I}}$ as the multiplier operator $T_{\mathcal{I}}$ given by*

$$\forall f \in L^1_{loc}(\mathbb{R}), \forall x \in \mathbb{R}, T_{\mathcal{I}} f(x,t) = \sum_{I \in \mathcal{I}} r_I(t) \langle f, h_I \rangle h_I(x), \text{ for some } t \in [0,1].$$

We write $T_{\mathcal{I}} := T_{\mathcal{I}}(.,t)$ for some fixed $t \in [0,1]$.

Remark : We have $\|S_{\mathcal{I}}f\|_p \approx \left(\int_0^1 \int_{\mathbb{R}} |T_{\mathcal{I}}f(x,t)|^p dxdt\right)^{\frac{1}{p}}$.

Proposition 6. *The multiplier operator $T_{\mathcal{I}}$ is self-adjoint, i.e. $T_{\mathcal{I}}^* = T_{\mathcal{I}}$.*

Proof :
Let $t \in [0,1]$. Let $(f,g) \in \mathcal{S}(\mathbb{R})^2$.

$$\begin{aligned}
\langle T_{\mathcal{I}}f, g\rangle &= \int_{\mathbb{R}} T_{\mathcal{I}}f(x)g(x)dx && (1)\\
&= \int_{\mathbb{R}} \sum_{I\in\mathcal{I}} r_I(t)\langle f, h_I\rangle h_I(x)g(x)dx && (2)\\
&= \int_{\mathbb{R}} \sum_{I\in\mathcal{I}} r_I(t)\left(\int_{\mathbb{R}} f(u)h_I(u)du\right) h_I(x)g(x)dx && (3)\\
&\underset{\text{Fubini}}{=} \int_{\mathbb{R}} \sum_{I\in\mathcal{I}} r_I(t)f(u)h_I(u)\left(\int_{\mathbb{R}} h_I(x)g(x)dx\right) du && (4)\\
&= \int_{\mathbb{R}} \sum_{I\in\mathcal{I}} r_I(t)f(u)h_I(u)\langle h_I, g\rangle du && (5)\\
&= \langle f, T_{\mathcal{I}}g\rangle && (6)
\end{aligned}$$

By uniqueness of the adjoint, we can conclude $T_{\mathcal{I}}^* = T_{\mathcal{I}}$.

2 An alternative proof for the boundedness of the square operator

This second part is the one my mentors wanted me to reach. In her thesis [CB15], Cristina Benea gave an alternative proof of the L^p boundedness of the square function for $p \in]1, \infty[$. My goal was to prove the boundedness of this operator, using the same methods, in the following cases :

- $L^1(\mathbb{R}) \longrightarrow L^{1,\infty}(\mathbb{R})$,
- $H^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$,
- $L^\infty(\mathbb{R}) \longrightarrow BMO(\mathbb{R})$.

Let's recall that, for a given finite collection $\mathcal{I}$ of dyadic intervals, the square function is defined by

$$S_{\mathcal{I}} f(x) = \left(\sum_{I \in \mathcal{I}} |\langle f, h_I \rangle|^2 |h_I(x)|^2 \right)^{\frac{1}{2}}.$$

And

$$\forall x \in \mathbb{R}, \forall I \in \mathcal{I}, |h_I(x)|^2 = \frac{1}{|I|}(1_{I_l}(x) - 1_{I_r}(x))^2 = \frac{1}{|I|} 1_I(x).$$

Hence

$$S_{\mathcal{I}} f(x) = \left(\sum_{I \in \mathcal{I}} |\langle f, h_I \rangle|^2 |h_I(x)|^2 \right)^{\frac{1}{2}} = \left(\sum_{I \in \mathcal{I}} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I(x) \right)^{\frac{1}{2}}.$$

2.1 The main cases

2.1.1 Cristina's proof: the case $L^p \longrightarrow L^p$ for $p \in]1, \infty[$

We give here the proof she wrote in [CB15], but with more details and explanations.
Let $\mathcal{I}$ be any collection of dyadic intervals.
We consider the square operator $S_{\mathcal{I}}$ associated with $\mathcal{I}$.
<u>The idea of the proof :</u>

- We want to estimate $\|S_{\mathcal{I}} f\|_p$ so we dualize the L^p norm : $\|S_{\mathcal{I}} f\|_p = \sup_{\|g\|_{p'} \leqslant 1} \int_{\mathbb{R}} S_{\mathcal{I}} f(x) g(x) dx$ (as $\mathcal{S}(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$, we just have to consider $g \in \mathcal{S}(\mathbb{R})$).
- We apply the lemma 2 by localizing the operator on "good" dyadic intervals I_0 on which we have the control on the averages of g and the L^2 norm of f in order to bound the quantities $\int_{\mathbb{R}} \widetilde{S}_{I_0} f(x) g(x) dx$, where $\widetilde{S_{I_0}}$ is the localized operator on I_0.

Let $I_0 \in \mathcal{D}$. Let $\mathfrak{I}$ be the collection of the dyadic intervals of $\mathcal{I}$ which are contained in I_0 : $\forall I \in \mathfrak{I}, I \in \mathcal{I}$ and $I \subset I_0$.
We set $\mathfrak{I}^+(I_0) = \{I' \in \mathcal{D} / \exists I \in \mathfrak{I}, I \subset I' \subset I_0\}$. and

$$\forall g \in \mathcal{S}(\mathbb{R}), size_{I_0}(g) = \sup\{\frac{1}{|3I'|} \int_{\mathbb{R}} 1_{3I'}(y)|g(y)|dy / I' \in \mathfrak{I}^+(I_0)\}.$$

⚠Let's recall that $3I'$ is defined as the dyadic interval with the same center than I' but three times larger than I'.
We begin with a lemma which focuses on localised square function

$$\widetilde{S_{I_0}}f(x) = \left(\sum_{I\in\mathfrak{I}} |\langle f, h_I\rangle|^2 |h_I(x)|^2\right)^{\frac{1}{2}}.$$

Lemma 2.

$$\forall f \in L^1_{loc}(\mathbb{R}), \forall g \in \mathcal{S}(\mathbb{R}), |\int_{\mathbb{R}} \widetilde{S_{I_0}}f(x)g(x)dx| \lesssim size_{I_0}(g)\frac{\|f1_{I_0}\|_2}{|I_0|^{\frac{1}{2}}}|I_0|.$$

Proof :
Let $g \in \mathcal{S}(\mathbb{R})$. Let $f \in L^1_{loc}(\mathbb{R})$.
We set $\mathfrak{J} = \{J \in \mathcal{D} / \forall I \in \mathfrak{I}, I \subsetneq 3J$ and J is maximal with this property$\}$.
We have the existence of $\mathfrak{J}$ because we consider a finite collection of dyadic intervals.

- We can easily see that they form a partition of $\mathbb{R}$ (thanks to the maximality condition associated to dyadic properties (propositon 1)). You can think about them as obtained by "Whitney" decompositions (see [GRAF108]).
  So

$$\int_{\mathbb{R}} \widetilde{S_{I_0}}f(x)g(x)dx = \sum_{J\in\mathfrak{J}} \int_J \widetilde{S_{I_0}}f(x)g(x)dx = \sum_{J\in\mathfrak{J}} \int_J \left(\sum_{I\in\mathfrak{I}} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}} g(x)dx.$$

  Let $J \in \mathfrak{J}$. If $\forall I \in \mathfrak{I}, J \cap I = \emptyset$, then thanks to the characteristic function appearing in the above expression, we don't count the term associated to this J in the summation.
  Otherwise, for all $I \in \mathfrak{I}$ such that $I \cap J \neq \emptyset$, thanks to properties of dyadic intervals we have $J \subset I$ or $I \subset J$. The last possibility is impossible by definition of J. So $J \subsetneq I$ and thus $|J| < |I|$.
  Hence, thanks to the triangle inequality, we get :

$$\begin{aligned} |\int_{\mathbb{R}} \widetilde{S_{I_0}}f(x)g(x)dx| &= |\sum_{J\in\mathfrak{J}} \int_J \left(\sum_{I\in\mathfrak{I}, |I|>|J|} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}} g(x)dx| && (7) \\ &\leqslant \sum_{J\in\mathfrak{J}} \int_J \left(\sum_{I\in\mathfrak{I}, |I|>|J|} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}} |g(x)|dx. && (8) \end{aligned}$$

- Now we focus on each integral.
  Let $J \in \mathfrak{J}$.
  $*$ As J is maximal with the property of no-inclusion, then $\tilde{J}$ isn't. So there exists $I_J \in \mathfrak{I}$ such that $I_J \subset 3\tilde{J}$. So $I_J \subset \tilde{J}$ or $I_J \subset \tilde{\tilde{J}}$ (or a translate of one unit). Hence, there exists $I' \in \mathcal{D}$ such that $I' \in \mathfrak{I}^+(I_0)$, $J \subset 3I'$ and $|J| < |I'| \leqslant 4|J|$.
  $*$ Let $I \in \mathfrak{I}$ such that $|I| > |J|$. If $I \cap J = \emptyset$, then, thanks to the characteristic function, we don't count the term associated to this I in the summation. Otherwise, we have $J \subsetneq I$ and then

$$\forall x \in J, \left(\sum_{I\in\mathfrak{I}, |I|>|J|} \frac{|\langle f, h_I\rangle|^2}{|I|} \underbrace{1_I(x)}_{=1}\right)^{\frac{1}{2}} = \left(\sum_{I\in\mathfrak{I}, |I|>|J|} \frac{|\langle f, h_I\rangle|^2}{|I|}\right)^{\frac{1}{2}} := C(J) \text{ constant.}$$

Moreover,

$$\forall x \in \mathbb{R}, C(J) = \left(\sum_{I\in\mathfrak{J}} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}} 1_J(x).(\star)$$

Hence,

$$\begin{aligned}
\int_J \left(\sum_{I\in\mathfrak{J},|I|>|J|} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}} |g(x)|dx &= \int_{\mathbb{R}} \frac{\overbrace{|3I'|}^{=3|I'|\leqslant 12|J|}}{|3I'|} \underbrace{1_J(x)}_{\leqslant 1_{3I'}(x)} C(J)|g(x)|dx && (9)\\
&\leqslant 12|J|C(J) \underbrace{\frac{1}{|3I'|}\int_{\mathbb{R}} 1_{3I'}(x)|g(x)|dx}_{\leqslant size_{I_0}(g)} && (10)\\
&\lesssim size_{I_0}(g)|J|C(J) && (11)\\
&\underset{(\star)}{\lesssim} size_{I_0}(g)\int_J \underbrace{\left(\sum_{I\in\mathfrak{J}} \frac{|\langle f, h_I\rangle|^2}{|I|} 1_I(x)\right)^{\frac{1}{2}}}_{=S_{\mathcal{I}}f(x)} dx && (12)
\end{aligned}$$

- Thanks to (8), (12) and the Cauchy-Schwarz inequality, we have (summing over J) :

$$|\int_{\mathbb{R}} \widetilde{S_{I_0}} f(x)g(x)dx| \lesssim size_{I_0}(g)\|S_{\mathcal{I}}f\|_2 \underbrace{\sqrt{|J|}}_{\leqslant\sqrt{|I_0|}} \underset{cfprop3}{\leqslant} size_{I_0}(g)\frac{\|f1_{I_0}\|_2}{|I_0|^{\frac{1}{2}}}|I_0|.$$

Theorem 3. *For all $p \in [2,\infty[$, $S_{\mathcal{I}}$ is of strong-type (p,p).*

Proof :
Let $p \in]1,\infty[$. Let $p' \in]1,\infty[$ such that $\frac{1}{p}+\frac{1}{p'}=1$.
It suffices to show thanks to restricted type interpolation (see [GRAF108]) that

$$\forall f \in L^1_{loc}(\mathbb{R}), \forall g \in \mathcal{S}(\mathbb{R}), \forall (F,G) \in \mathcal{B}(\mathbb{R})^2, (|f| \leqslant 1_F \text{ and } |g| \leqslant 1_G) \Rightarrow |\int_{\mathbb{R}} S_{\mathcal{I}}f(x)g(x)dx| \lesssim |F|^{\frac{1}{p}}|G|^{\frac{1}{p'}}.$$

The previous lemma shows us that, to prove the boundedness of the operator $S_{\mathcal{I}}$, it suffices to control the bound on g (through $size_{I_0}(g)$) and the one on f (through $\|f1_{I_0}\|_2$).
We use for that a stopping-time argument.

- First, we focus on f.
  $*$ At step k , we set

$$\mathcal{I}_{n_k} = \left\{ I_0 \in \mathcal{D} / \begin{array}{l} \exists I \in \mathcal{I}, I \subset I_0, \forall i \in [\![1,k-1]\!], I_0 \notin \mathcal{I}_{n_i} \\ 2^{-n_k-1} \leqslant \frac{1}{|I_0|}\int_{\mathbb{R}} 1_F(x)1_{I_0}(x)dx \leqslant 2^{-n_k} \\ \forall I' \subset I_0, (\exists I \in \mathcal{I}, I \subset I') \Rightarrow \frac{1}{|I'|}\int_{\mathbb{R}} 1_F(x)1_{I'}(x)dx \leqslant 2^{-n_k} \\ I_0 \text{ maximal with this property} \end{array} \right\}.$$

  The reader can take $n_k = k \in \mathbb{N}$ even if it means that $\mathcal{I}_{n_k} = \emptyset$. Actually, we choose $n_{k+1} = \min\{n \in \mathbb{N}\backslash[\![0,n_k]\!] / \mathcal{I}_n \neq \emptyset\}$.
  The purpose is to get intervals which control the boundedness in f and taken with maximal and optimal conditions. Thus, we build a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and the associated sequence $(\mathcal{I}_{n_k})_{k\in\mathbb{N}}$ until we exhaust all the intervals in $\mathcal{I}$ (which one is countable so the stopping-time argument can be used).

$*$ Let $k \in \mathbb{N}$. Let $I_0 \in \mathcal{I}_{n_k}$.
We have

$$2^{-n_k-1} \leqslant \frac{1}{|I_0|} \int_{\mathbb{R}} 1_F(x) 1_{I_0}(x) dx \Leftrightarrow \frac{|I_0|}{2} \leqslant 2^{n_k} \int_{I_0} 1_F(x) dx.$$

Summing over all I_0 in $\mathcal{I}_{n_k}$, as they are pairewise disjoint thanks to their maximal condition, we have :

$$\frac{1}{2} \sum_{I_0 \in \mathcal{I}_{n_k}} |I_0| \leqslant 2^{n_k} \sum_{I_0 \in \mathcal{I}_{n_k}} \int_{I_0} 1_F(x) dx = 2^{n_k} \int_{\mathbb{R}} 1_F(x) dx = 2^{n_k} |F|.$$

Thus,

$$\boxed{\sum_{I_0 \in \mathcal{I}_{n_k}} |I_0| \lesssim 2^{n_k} |F|} \ (\star)$$

- Now we focus on g.
  $*$ At step l, we want to control $size(g)$ for intervals in the collection $\mathcal{I}$.
  We choose $M_l \in \mathcal{I}$ such that M_l is maximal with the property

$$\frac{1}{|3M_l|} \int_{\mathbb{R}} 1_{3M_l}(x) |g(x)| dx = \sup_{I \in \mathcal{I}} \frac{1}{|3I|} \int_{\mathbb{R}} 1_{3I}(x) |g(x)| dx \leqslant 2^{-m_l}.$$

(The sup is a max because we consider a finite collection $\mathcal{I}$, so such an interval exists).
We set

$$\mathcal{I}_{m_l} = \left\{ I_0' \in \mathcal{D} / \begin{array}{c} M_l \subset I_0', \forall i \in [\![1, l-1]\!], I_0' \notin \mathcal{I}_{m_i} \\ 2^{-m_l - 1} \leqslant \frac{1}{|3I_0'|} \int_{\mathbb{R}} 1_{3I_0'}(x) |g(x)| dx \leqslant 2^{-m_l} \\ I_0' \text{ maximal with this property} \end{array} \right\}.$$

One more time, the reader can take $m_l = l$.
The purpose is to get intervals which control the boundedness in g and taken with maximal and optimal conditions. Thus, we build a strictly increasing sequence $(m_l)_{l \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and the associated sequence $(\mathcal{I}_{m_l})_{l \in \mathbb{N}}$ until we exhaust all the intervals in $\mathcal{I}$.
$*$ For all $l \in \mathbb{N}$, the $I_0' \in \mathcal{I}_{m_l}$ are pairewise disjoint thanks to their maximal condition.
Because of the sup, we can't conclude directly to an equality of the form $(\star)$.
We need that the Hardy-Littlewood maximal operator is of weak-type $(1,1)$:

$$\begin{aligned} \forall l \in \mathbb{N}, \sum_{I_0' \in \mathcal{I}_{m_l}} |I_0'| &= |\bigsqcup_{I_0' \in \mathcal{I}_{m_l}} I_0'| && (13) \\ &\leqslant |\{x \in \mathbb{R} / Mg(x) > 2^{-m_l}\}| && (14) \\ &\leqslant |\{x \in \mathbb{R} / M1_G(x) > 2^{-m_l}\}| && (15) \\ &\lesssim 2^{m_l} \|1_G\|_1 && (16) \\ &= 2^{m_l} |G|. && (17) \end{aligned}$$

Thus

$$\boxed{\sum_{I_0' \in \mathcal{I}_{m_l}} |I_0'| \lesssim 2^{m_l} |G|} \ (\star)$$

- We need to dualize the l^2 norm (knowing that $(l^2)^* = l^2$) :

$$\left(\sum_{I \in \mathcal{I}} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I(x) \right)^{\frac{1}{2}} = \sum_{I \in \mathcal{I}} \varepsilon_I(x) \langle f, h_I \rangle h_I(x) \text{ with } \left(\sum_{I \in \mathcal{I}} \varepsilon_I(x) \right)^{\frac{1}{2}} = 1 \text{ for a.e. } x \in \mathbb{R}.$$

Then, localizing we have :

$$\int_{\mathbb{R}} S_{\mathcal{I}}f(x)g(x)dx = \int_{\mathbb{R}} \sum_{I\in\mathcal{I}} \varepsilon_I(x)\langle f,h_I\rangle h_I(x)g(x)dx \tag{18}$$

$$= \sum_{(k,l)\in\mathbb{N}^2} \sum_{I_0\in\mathcal{I}_{n_k}\cap\mathcal{I}_{m_l}} \int_{\mathbb{R}} \sum_{I\in\mathcal{I},I\subset I_0} \varepsilon_I(x)\langle f,h_I\rangle h_I(x)g(x)dx. \tag{19}$$

Then, the lemma 2 and the definitions of $\mathcal{I}_{n_k}$ and $\mathcal{I}_{m_l}$ give :

$$|\int_{\mathbb{R}} S_{\mathcal{I}}f(x)g(x)dx| \leqslant \sum_{(k,l)\in\mathbb{N}^2} \sum_{I_0\in\mathcal{I}_{n_k}\cap\mathcal{I}_{m_l}} |\langle \widetilde{S_{I_0}}f,g\rangle| \lesssim \sum_{(k,l)\in\mathbb{N}^2} \sum_{I_0\in\mathcal{I}_{n_k}\cap\mathcal{I}_{m_l}} 2^{\frac{-n_k}{2}}2^{-m_l}|I_0|.$$

⚠The $\frac{1}{2}$ in the power of 2 comes from the L^2 norm of the characteristic function of F appearing in lemma 2 whereas we considered only the L^1 norm in the beginning of this proof. The L^2 and L^1 norm of a characteristic function being very close, we only get this $\frac{1}{2}$.
Taking a geometric average of $(\star)$ and $(\star)$ we have for all $(\theta_1,\theta_2)\in(\mathbb{R}_+^*)^2$ such that $\theta_1+\theta_2=1$:

$$|\int_{\mathbb{R}} S_{\mathcal{I}}f(x)g(x)dx| \lesssim \sum_{(k,l)\in\mathbb{N}^2} 2^{\frac{-n_k}{2}}2^{-m_l}(2^{n_k}|F|)^{\theta_1}(2^{m_l}|G|)^{\theta_2} = \sum_{(k,l)\in\mathbb{N}^2} 2^{-n_k(\frac{1}{2}-\theta_1)}2^{-m_l(1-\theta_2)}|F|^{\theta_1}|G|^{\theta_2}.$$

To reach our goal, we take $\theta_1=\frac{1}{p}$ and $\theta_2=\frac{1}{p'}$.
⚠To still have a convergent serie, we must have $\frac{1}{2}-\frac{1}{p}>0$ which implies $p>2$.
The case $p=2$ has already been proved in the first part of this internship report.

Now we prove the L^p boundedness for $p\in]1,\infty[$ looking at the linearization of $S_{\mathcal{I}}$, we denoted by $T_{\mathcal{I}}$ (recall that t is fixed in $[0,1]$).

$$\forall f\in L^1_{loc}(\mathbb{R}),\forall x\in\mathbb{R}, T_{\mathcal{I}}f(x)=\sum_{I\in\mathcal{I}} r_I(t)\langle f,h_I\rangle h_I(x).$$

Theorem 4. *For all $p\in]1,\infty[$, $T_{\mathcal{I}}$ is of strong-type (p,p).*

Proof :
We take the proof of the lemma 2. It's identical until the use of the Cauchy-Schwarz inequality.
We just have to prove the L^2 boundedness of $T_{\mathcal{I}}$.
Let $f\in L^1_{loc}(\mathbb{R})$.
We have, writing $\widetilde{T_{I_0}}$ the localized operator,

$$\|\widetilde{T_{I_0}}f\|_2 = \langle\sum_{I\in\mathfrak{I}} r_I(t)\langle f,h_I\rangle,\sum_{I'\in\mathfrak{I}} r_{I'}(t)\langle f,h_{I'}\rangle h_{I'}\rangle \tag{20}$$

$$\overset{\perp}{=} \sum_{I\in\mathfrak{I}} |\langle f,h_I\rangle|^2 \tag{21}$$

$$= \langle f,\sum_{I\in\mathfrak{I}}\langle f,h_I\rangle h_I\rangle \tag{22}$$

$$\underset{\text{Cauchy-Schwarz}}{\leqslant} \|f\|_2\|\sum_{I\in\mathfrak{I}}\langle f,h_I\rangle h_I\|_2 \tag{23}$$

$$\overset{\perp}{=} \|f\|_2\left(\sum_{I\in\mathfrak{I}}|\langle f,h_I\rangle|^2\right)^{\frac{1}{2}} \tag{24}$$

$$\underset{\text{Bessel}}{\leqslant} \|f\|_2^2 \tag{25}$$

Then the proof of the lemma 2 is done for $T_{\mathcal{I}}$. The boundedness for $p \geqslant 2$ is identical.
Finally, as $T_{\mathcal{I}}$ is self-adjoint (proposition 6), we can conculde by duality ($\forall p \in]1,\infty[, (L^p(\mathbb{R}))^* = L^{p'}(\mathbb{R})$) that $T_{\mathcal{I}}$ is of strong type (p,p) for all $p \in]1,\infty[$.

2.1.2 Case $L^1 \longrightarrow L^{1,\infty}$

Now we want to get the previous result also for $p = 1$. The theory gives that actually $S_{\mathcal{I}}$ (or $T_{\mathcal{I}}$) is of weak-type $(1,1)$. Our job is to prove this result using the previous method (and not the "classical" one based on the Calderon-Zygmund decomposition).

We have to see where are the problems in the previous proofs in order to defuse it.

- The previous proofs give that the operator is of strong-type, but we want it to be of weak-type.
  To prove the boundedness in L^p we used the dualization of the L^p norm. Here we need the dualization of the weak-L^1 semi-norm if we want to use the same method.

- In the last formular of the theorem 3, we can't take $\theta_2 = 0$ (i.e. $p' = \infty$, i.e. $p = 1$). Otherwise we would get a non-summable serie.

As noted in the first remark above, we need to dualise the semi-norm $\|.\|_{1,\infty}$. This result is given by the following lemma (see [GRAF108]) :

Lemma 3 (Dualization of the semi-norm $\|.\|_{1,\infty}$)**.**

$$\|f\|_{L^{1,\infty}(\mathbb{R})} \approx \sup_{E\subset\mathbb{R},0<|E|<\infty} \inf_{E'\subset E,|E'|\geqslant\frac{1}{2}|E|} |\int_{E'} f(x)dx|.$$

Now, we see that we have to estimate the quantities $\displaystyle\int_{\mathbb{R}} T_{\mathcal{I}}f(x)1_{E'}(x)dx = \langle T_{\mathcal{I}}f, 1_{E'}\rangle$.

Theorem 5. *$T_{\mathcal{I}}$ is of weak-type $(1,1)$.*

Proof :

- Using the self-adjointness of the operator $T_{\mathcal{I}}$ and the lemma 2, we have

$$\forall f \in \mathcal{S}(\mathbb{R}), \forall E' \subset \mathbb{R}, |\langle T_{\mathcal{I}}f, 1_{E'}\rangle| = |\langle f, T_{\mathcal{I}}1_{E'}\rangle| \lesssim size_{I_0}(f)\frac{\|1_{E'}1_{I_0}\|_2}{|I_0|^{\frac{1}{2}}}|I_0|.$$

  The idea is to exchange the roles of f and $g = 1_{E'}$, thanks to duality, in order to get back the case $p = 1$ as we did in proposition 7 to get the case $p \in]1,2[$.

- Let $f \in \mathcal{S}(\mathbb{R})$. Thanks to the linearity of $T_{\mathcal{I}}$, we can suppose $\|f\|_1 = 1$.
  Let $E \subset \mathbb{R}$ such that $0 < |E| < \infty$. We can assume $|E| = 1$.
  We set

$$\Omega = \{x \in \mathbb{R}/Mf(x) > C\} \text{ and } E' = E\backslash\Omega.$$

  C will be chosen correctly (high enough) such that $|E'| \geqslant \frac{1}{2}|E|$.
  For all $d \in \mathbb{N}$, we set $\mathcal{I}_d = \{I \in \mathcal{D}/1 + \frac{dist(I,\Omega^c)}{|I|} \approx 2^d\}$.
  Now we perform the stopping time argument as in the original proof at each fixed $d \in \mathbb{N}$.
  Let $d \in \mathbb{N}$.

$*$ We focus on f :
At step k, we choose $M_k \in \mathcal{I}$ such that M_k is maximal with the property

$$\frac{1}{|3M_k|}\int_{\mathbb{R}} 1_{3M_k}(x)|f(x)|dx = \sup_{I\in\mathcal{I}} \frac{1}{|3I|}\int_{\mathbb{R}} 1_{3I}(x)|f(x)|dx \leqslant 2^{-n_k}$$

where we choose the largest possible value of 2^{-n_k}.
(The sup is a max because we consider a finite collection $\mathcal{I}$, so such an interval exists).
We set

$$\mathcal{I}^d_{n_k} = \left\{ I'_0 \in \mathcal{D}\cap\mathcal{I}_d / \begin{array}{c} M_k \subset I'_0, \forall i \in [\![1, k-1]\!], I'_0 \notin \mathcal{I}^d_{n_i} \\ 2^{-n_k-1} \leqslant \frac{1}{|3I'_0|}\int_{\mathbb{R}} 1_{3I'_0}(x)|f(x)|dx \leqslant 2^{-n_k} \\ I'_0 \text{ maximal with this property} \end{array} \right\}.$$

Thus, we build a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ and the associated sequence $(\mathcal{I}_{n_k})_{k\in\mathbb{N}}$ until we exhaust all the intervals in $\mathcal{I}$.
Thanks to the fact that the Hardy-Littlewood maximal operator is of weak-type $(1,1)$, we have :

$$\begin{aligned} \forall k \in \mathbb{N}, \sum_{I'_0\in\mathcal{I}^d_{n_k}} |I'_0| &= |\bigsqcup_{I'_0\in\mathcal{I}^d_{n_k}} I'_0| && (26)\\ &\leqslant |\{x \in \mathbb{R}/Mf(x) > 2^{-n_k}\}| && (27)\\ &\lesssim 2^{n_k}\|f\|_1 && (28)\\ &= 2^{n_k}. && (29) \end{aligned}$$

Thus,

$$\boxed{\sum_{I'_0\in\mathcal{I}^d_{n_k}} |I'_0| \lesssim 2^{n_k}}$$

Moreover, let $I'_0 \in \mathcal{I}^d_{n_k}$. By definition of I'_0, we have that

$$2^d I'_0 \cap \Omega^c \neq \emptyset.$$

(Actually, $\mathcal{I}_{n_k}$ needn't be in $\mathcal{I}_d$ in order that the previous formula holds (and that solves a possible problem of existence)).
So there exists $x_0 \in 2^d I'_0 \cap \Omega^c$. We have :

$$Mf(x_0) \leqslant C \text{ and so } \frac{1}{|2^d I'_0|}\int_{2^d I'_0} |f(x)|dx \leqslant C.$$

We choosed I'_0 such that :

$$\frac{1}{|3I'_0|}\int_{3I'_0} |f(x)|dx \approx 2^{-n_k}$$

But, as $3I'_0 \subset 2^d I'_0$ (for $d \geqslant 2$), we have :

$$2^{-n_k} \lesssim \frac{1}{|3I'_0|}\int_{3I'_0} |f(x)|dx \leqslant \frac{1}{|3I'_0|}\int_{2^dI'_0} |f(x)|dx \lesssim 2^d\frac{1}{|2^dI'_0|}\int_{2^dI'_0} |f(x)|dx \leqslant 2^d C \lesssim 2^d.$$

If $d = 0$ (resp. $d = 1$), we have $I \subset 4I$ (resp. $2I \subset 4I$).
As $I \in \mathcal{I}_0$ (resp. $I \in \mathcal{I}_1$), we have $I \cap \Omega^c \neq \emptyset$ (resp. $2I \cap \Omega^c \neq \emptyset$).
So $4I \cap \Omega^c \neq \emptyset$ and we have the same result.
Hence we have :

$$\boxed{2^{-n_k} \lesssim 2^d}$$

Remark : This last expression also states that we will still have a convergent serie, because, even if this time $n_k \in \mathbb{Z}$, it can't be "too much" negative and thus we can sum.
$*$ Now we focus on $1_{E'}$:
At step l , we set

$$\mathcal{I}_{m_l}^d = \left\{ I_0 \in \mathcal{D} \cap \mathcal{I}_d / \begin{array}{l} \exists I \in \mathcal{I}, I \subset I_0, \forall i \in [\![1, l-1]\!], I_0 \notin \mathcal{I}_{m_i}^d \\ 2^{-m_l-1} \leqslant \frac{1}{|I_0|} \int_{\mathbb{R}} 1_{E'}(x) 1_{I_0}(x) dx \leqslant 2^{-m_l} \\ \forall I' \subset I_0, (\exists I \in \mathcal{I}, I \subset I') \Rightarrow \frac{1}{|I'|} \int_{\mathbb{R}} 1_{E'}(x) 1_{I'}(x) dx \leqslant 2^{-m_l} \\ I_0 \text{ maximal with this property} \end{array} \right\}.$$

Thus, we build a strictly increasing sequence $(m_l)_{l \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and the associated sequence $(\mathcal{I}_{m_l})_{l \in \mathbb{N}}$ until we exhaust all the intervals in $\mathcal{I}$
We still have

$$\frac{1}{2} \sum_{I_0 \in \mathcal{I}_{m_l}^d} |I_0| \leqslant 2^{m_l} \sum_{I_0 \in \mathcal{I}_{m_l}^d} \int_{I_0} 1_{E'}(x) dx \leqslant 2^{m_l} \int_{\mathbb{R}} 1_{E'}(x) dx = 2^{m_l} |E'| \lesssim 2^{m_l}.$$

Thus,

$$\boxed{\sum_{I_0 \in \mathcal{I}_{m_l}^d} |I_0| \lesssim 2^{m_l}}$$

We know that $supp(1_{E'}) = E' \subset \Omega^c$. Let $l \in \mathbb{N}$.
If $I_0' \in \mathcal{I}_{m_l}^d$, then $I_0' \cap E' \neq \emptyset$. So $I_0' \cap \Omega^c \neq \emptyset$. So $dist(I_0', \Omega^c) = 0$. So $1 \approx 2^d$. So $d = 0$.
Hence

$$\boxed{2^{-m_l} \lesssim 2^{-Md}}$$

for an arbitrary $M > 2$.

- Thus, we have :

$$\begin{aligned}
|\langle T_{\mathcal{I}} f, 1_{E'} \rangle| &\leqslant \sum_{d=0}^{\infty} \sum_{(k,l) \in \mathbb{N}^2} \sum_{I_0 \in \mathcal{I}_{n_k}^d \cap \mathcal{I}_{m_l}^d} |\langle \widetilde{T_{I_0}} f, 1_{E'} \rangle| && (30) \\
&\lesssim \sum_{d=0}^{\infty} \sum_{(k,l) \in \mathbb{N}^2} \sum_{I_0 \in \mathcal{I}_{n_k}^d \cap \mathcal{I}_{m_l}^d} size_{I_0}(f) \frac{\|1_{E'} 1_{I_0}\|_2}{|I_0|^{\frac{1}{2}}} |I_0| && (31) \\
&\lesssim \sum_{d=0}^{\infty} \sum_{(k,l) \in \mathbb{N}^2} 2^{-n_k} 2^{-\frac{m_l}{2}} \sum_{I_0 \in \mathcal{I}_{n_k}^d \cap \mathcal{I}_{m_l}^d} |I_0| && (32) \\
&\lesssim \sum_{d=0}^{\infty} \sum_{(k,l) \in \mathbb{N}^2} 2^{-n_k} 2^{-\frac{m_l}{2}} 2^{\frac{2n_k}{3}} 2^{\frac{m_l}{3}} && (33) \\
&= \sum_{d=0}^{\infty} \sum_{(k,l) \in \mathbb{N}^2} 2^{-\frac{n_k}{3}} 2^{-\frac{m_l}{6}} && (34) \\
&\lesssim \sum_{d=0}^{\infty} 2^{\frac{d}{3}} 2^{-\frac{Md}{6}} && (35) \\
&= \left(\sum_{d=0}^{\infty} 2^{\frac{d}{6}(2-M)} \right) \underbrace{\|f\|_1}_{=1} && (36)
\end{aligned}$$

Remark : Only the term corrsponding to $d = 0$ is present, but we chosed to let the serie appearing because in the case of general wevelets (different from the Haar system) this is the point.
Hence,

$$\forall f \in L^1(\mathbb{R}), \|T_{\mathcal{I}} f\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})}$$

i.e. $T_{\mathcal{I}} : L^1(\mathbb{R}) \longrightarrow L^{1,\infty}(\mathbb{R})$ is bounded.

2.2 Cases $H^1 \longrightarrow L^1$ and $L^\infty \longrightarrow BMO$

2.2.1 Case $H^1 \longrightarrow L^1$

Theorem 6. *$T_{\mathcal{I}} : H^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$ is bounded.*

Proof :
Let $f \in H^1(\mathbb{R})$.

There exists $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in l^1(\mathbb{N}, \mathbb{C})$ and $(a_i)_{i\in\mathbb{N}}$ a sequence of atoms such that $f = \displaystyle\sum_{i=0}^{\infty} \lambda_i a_i$.

Assume for now that we have :

$$T_{\mathcal{I}} f = \sum_{i=0}^{\infty} \lambda_i T_{\mathcal{I}} a_i \ (\star)$$

So by the triangle inequality we have :

$$\|T_{\mathcal{I}} f\|_1 \leqslant \sum_{i=0}^{\infty} |\lambda_i| \|T_{\mathcal{I}} a_i\|_1.$$

So we have to estimate the quantities $\|T_{\mathcal{I}} a_i\|_1$.
Let $i \in \mathbb{N}$. We set $g(x) = 1_{T_{\mathcal{I}} a_i \geqslant 0}(x) - 1_{T_{\mathcal{I}} a_i < 0}(x)$.

$$\|T_{\mathcal{I}} a_i\|_1 = \int_{\mathbb{R}} |T_{\mathcal{I}} a_i(x)| dx = \langle T_{\mathcal{I}} a_i, g\rangle = \sum_{I\in\mathcal{I}} r_I(t) \langle f, h_I\rangle \langle h_I, g\rangle.$$

Let Q_i be as in the definition of the Hardy space :

- $supp(a_i) \subset Q_i$ (1)
- $\|a_i\|_2 \leqslant \frac{1}{\sqrt{|Q_i|}}$ (2)
- $\displaystyle\int_{Q_i} a_i(x) dx = 0$ (3)

We look at $\langle a_i, h_I\rangle \underset{(1)}{=} \displaystyle\int_{Q_i} a_i(x) h_I(x) dx$.
∗ If $I \cap Q_i = \emptyset$, then $\langle a_i, h_I\rangle = 0$. So we can not count those intervals in the summation.
∗ Otherwise, we have $Q_i \subsetneq I$ or $I \subset Q_i$ (dyadic properties).
If $Q_i \subsetneq I$, then $Q_i \subset I_l$ or $Q_i \subset I_r$ (dyadic properties).
We can suppose by symmetry that $Q_i \subset I_l$.
Then :

$$\langle a_i, h_I\rangle = \int_{I_l} \frac{a_i(x)}{\sqrt{|I|}} dx = \frac{1}{\sqrt{|I|}} \int_{Q_i} a_i(x) dx \underset{(3)}{=} 0.$$

So we don't count those intervals in the summation.
Thus we only have to consider the localized operator $\widetilde{T_{Q_i}}$ on the interval Q_i.

$$\begin{aligned}
\|T_{\mathcal{I}} a_i\|_1 &= \|\widetilde{T_{Q_i}} a_i\|_1 && (37)\\
&= \langle \widetilde{T_{Q_i}} a_i, g 1_{Q_i}\rangle && (38)\\
&\underset{\text{lemma 2}}{\lesssim} \underbrace{size_{Q_i}(g)}_{\leqslant 2} \frac{\|a_i 1_{Q_i}\|_2}{|Q_i|^{\frac{1}{2}}} |Q_i| && (39)\\
&\lesssim \frac{\|a_i\|_2}{|Q_i|^{\frac{1}{2}}} |Q_i| && (40)\\
&\underset{(2)}{\leqslant} 1 && (41)
\end{aligned}$$

Finally,

$$\|T_{\mathcal{I}} f\|_1 \lesssim \sum_{i=0}^{\infty} |\lambda_i| = \|\lambda\|_{l^1(\mathbb{N},\mathbb{C})}.$$

Taking the infimum, we get

$$\|T_{\mathcal{I}} f\|_1 \lesssim \|f\|_{H^1}.$$

But we stil have to prove $(\star)$.
Let $\alpha > 0$. Let $n \in \mathbb{N}$.
As $T_{\mathcal{I}} : L^1(\mathbb{R}) \longrightarrow L^{1,\infty}(\mathbb{R})$ is bounded, we have :

$$\begin{aligned}
& |\{x \in \mathbb{R} / |T_{\mathcal{I}} f(x) - \sum_{i=0}^{\infty} \lambda_i T_{\mathcal{I}} a_i(x)| > \alpha\}| \\
\leqslant\ & |\{x \in \mathbb{R} / |T_{\mathcal{I}} f(x) - \sum_{i=0}^{n} \lambda_i T_{\mathcal{I}} a_i(x)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R} / |\sum_{i=n+1}^{\infty} \lambda_i T_{\mathcal{I}} a_i(x)| > \frac{\alpha}{2}\}| \\
\leqslant\ & \frac{2}{\alpha} \|T_{\mathcal{I}}\|_{1\to(1,\infty)} \|f - \sum_{i=0}^{n} \lambda_i a_i\|_1 + \frac{2}{\alpha} \|\sum_{i=n+1}^{\infty} \lambda_i T_{\mathcal{I}} a_i\|_1 \\
\leqslant\ & \frac{2}{\alpha} \|T_{\mathcal{I}}\|_{1\to(1,\infty)} \|f - \sum_{i=0}^{n} \lambda_i a_i\|_1 + \frac{2}{\alpha} \sum_{i=n+1}^{\infty} |\lambda_i| \underbrace{\|T_{\mathcal{I}} a_i\|_1}_{\lesssim 1} \\
\lesssim\ & \|f - \sum_{i=0}^{n} \lambda_i a_i\|_1 + \sum_{i=n+1}^{\infty} |\lambda_i|.
\end{aligned}$$

But as $f = \displaystyle\sum_{i=0}^{\infty} \lambda_i a_i$ in $H^1(\mathbb{R})$, $\left(\displaystyle\sum_{i=0}^{n} \lambda_i a_i\right)_{n\in\mathbb{N}}$ converges to f in $L^1(\mathbb{R})$.
Moreover, as $\lambda \in l^1(\mathbb{N},\mathbb{C})$, $\left(\displaystyle\sum_{i=n+1}^{\infty} |\lambda_i|\right)_{n\in\mathbb{N}}$ converges to 0.
Thus,

$$\forall \alpha > 0, |\{x \in \mathbb{R} / |T_{\mathcal{I}} f(x) - \sum_{i=0}^{\infty} \lambda_i T_{\mathcal{I}} a_i(x)| > \alpha\}| = 0.$$

Hence

$$T_{\mathcal{I}} f = \sum_{i=0}^{\infty} \lambda_i a_i \text{ a.e.}$$

2.2.2 Case $L^\infty \longrightarrow BMO$

Theorem 7. *$T_{\mathcal{I}} : L^\infty(\mathbb{R}) \longrightarrow BMO(\mathbb{R})$ is bounded.*

Proof :
As

- $(H^1(\mathbb{R}))^* = BMO(\mathbb{R})$,
- $(L^1(\mathbb{R}))^* = L^\infty(\mathbb{R})$,
- $T_{\mathcal{I}}^* = T_{\mathcal{I}}$,
- $T_{\mathcal{I}} : H^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$ is bounded,

we have : $T_{\mathcal{I}} : L^\infty(\mathbb{R}) \longrightarrow BMO(\mathbb{R})$ is bounded.

Final remark : In the case L^p for $p \in]1, \infty[$, it wasn't rectrive to work with a finite collection $\mathcal{I}$ of dyadic intervals.
Indeed, as $\mathcal{D} = \{[k2^n, (k+1)2^n[/(k,n) \in \mathbb{Z}^2\}$, an infinite collection of dyadic intervals is a countable set. Let $\mathcal{I}_\infty$ be an infinite collection dyadic intervals :

$$\mathcal{I}_\infty := \{I_j, j \in \mathbb{N}\}.$$

Let $f \in L^p(\mathbb{R})$.
For all $j \in \mathbb{N}$, we set $\mathcal{I}_j := \{I_k / k \in [\![0, j]\!]\}$ and $f_j := S_{\mathcal{I}_j}(f) \in L^p(\mathbb{R})$.
$\forall j \in \mathbb{N}, f_j \geqslant 0$ and $(f_j)_{j \in \mathbb{N}}$ is an increasing sequence which converges a.e. to $f_\infty := S_{\mathcal{I}_\infty}(f)$.
Moreover, $\forall j \in \mathbb{N}, \|f_j\|_p \leqslant C\|f\|_p$ with C independant of j and f.
Hence, by the monotone convergence theorem, $\|f_\infty\|_p \leqslant M\|f\|_p$ and $S_{\mathcal{I}_\infty} : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ is bounded.

3 The $T(1)$ theorem

3.1 Singular integral operators

3.1.1 Calderon-Zygmund operators

Definition 13. *Let $\alpha \in]0,1]$. We call Calderon-Zygmund kernel of order α a continuous function $K : \Delta^c \to \mathbb{C}$ such that there exist $C > 0$ such that :*

- $\forall (x,y) \in^c \Delta, |K(x,y)| \leqslant \frac{C}{|x-y|^n}$.
- $\forall (x,y,y') \in (\mathbb{R}^n)^3, (2|y-y'| \leqslant |x-y|$ *and* $x \neq y) \Rightarrow |K(x,y)-K(x,y')| \leqslant C(\frac{|y-y'|}{|x-y|})^\alpha \frac{1}{|x-y|^n}$.
- $\forall (x,x',y) \in (\mathbb{R}^n)^3, (2|x-x'| \leqslant |x-y|$ *and* $x \neq y) \Rightarrow |K(x,y)-K(x',y)| \leqslant C(\frac{|x-x'|}{|x-y|})^\alpha \frac{1}{|x-y|^n}$.

We write $K \in CZK_\alpha$.

Remark : *These are technical conditions of smoothness and such a kernel present a singularity on $\{x = y\}$.*

Definition 14. *Let $\alpha \in]0,1]$. Let $K \in CZK_\alpha$. Let $T \in \mathcal{L}(L^2(\mathbb{R}^n))$ such that*

$$\forall f \in \mathcal{S}(\mathbb{R}^n), Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \text{ for a.e. } x \in \mathbb{R}^n.$$

We say that T is a Calderon-Zygmund operator of order α associated to $K \in CZK_\alpha$, and we write $T \in CZO_\alpha$.

3.1.2 Singular integral operators

Definition 15. *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be an operator. We say that T is a singular integral operator if its Schwartz kernel K defined by*

$$K \in \mathcal{S}(\mathbb{R}^{2n}) \text{ and, } \langle K, g \otimes f \rangle = \langle Tf, g \rangle$$

when restricted to Δ^c belongs to CZK_α.
We write $T \in SIO$.

3.2 Two useful lemmas

3.2.1 Cotlar's lemma

Lemma 4 (Cotlar's lemma)**.** *Let H be a Hilbert space. Let $N \in \mathbb{N}$. Let $(T_i)_{1 \leqslant i \leqslant N} \in \mathcal{L}(H)^N$. Let $\gamma \in l^1(\mathbb{Z}, \mathbb{R}_+)$.*
We assume that

$$\forall (j,k) \in [\![1,N]\!], \|T_j^* T_k\| \leqslant \gamma(j-k)^2 \text{ and } \|T_j T_k^*\| \leqslant \gamma(j-k)^2.$$

Then

$$\|\sum_{i=1}^N T_i\| \leqslant \|\gamma\|_{l^1(\mathbb{Z},\mathbb{R}_+)}.$$

Proof :

We set $T = \sum_{i=1}^{N} T_i \in \mathcal{L}(H)$.
Since T^*T is self-adjoint, one has $\|(T^*T)^m\| = \|T^*T\|^m = \|T\|^{2m}$.
But $(T^*T)^m = \sum_{1\leqslant i_1,\dots,i_m,j_1,\dots,j_m\leqslant N} \prod_{k=1}^{m} T^*_{i_k}T_{j_k}$.
By ordering the terms, we obtain both :

$$\|\prod_{k=1}^{m} T^*_{i_k}T_{j_k}\| \leqslant \prod_{k=1}^{m} \gamma(i_k - j_k)^2$$

and

$$\|\prod_{k=1}^{m} T^*_{i_k}T_{j_k}\| \leqslant \|T_{i_1}\|\|T_{j_m}\| \prod_{k=1}^{m-1} \gamma(j_k - i_{k+1})^2.$$

Taking the geometric average, we have :

$$\|\prod_{k=1}^{m} T^*_{i_k}T_{j_k}\| \leqslant (\|T_{i_1}\|\|T_{j_m}\|)^{\frac{1}{2}} \prod_{k=1}^{m} \gamma(i_k - j_k) \prod_{l=1}^{m} \gamma(j_k - i_{k+1}).$$

Then

$$\|T\|^{2m} \leqslant N \sup\{\|T_i\|/1 \leqslant i \leqslant N\} \|\gamma\|^{2m}_{l^1(\mathbb{Z},\mathbb{R}_+)}.$$

So

$$\|T\| \leqslant (N \sup\{\|T_i\|/1 \leqslant i \leqslant N\})^{\frac{1}{2m}} \|\gamma\|_{l^1(\mathbb{Z},\mathbb{R}_+)}.$$

Finaly, when m tends to $+\infty$,

$$\|T\| \leqslant \|\gamma\|_{l^1(\mathbb{Z},\mathbb{R}_+)}.$$

3.2.2 Schur's lemma

Lemma 5 (Schur's lemma)**.** *Let X and Y be spaces. We consider a positive mesure $\mu \otimes \nu$ on $X \times Y$. Let $K : X \times Y \to \mathbb{R}$ be a mesurable function. We define*

$$(Tf)(x) = \int_Y K(x,y)f(y)\nu(dy).$$

Then

- $\|T\|_{1\to 1} \leqslant \sup\{\int_X |K(x,y)|\mu(dx)/y \in Y\} = A$.
- $\|T\|_{\infty\to\infty} \leqslant \sup\{\int_Y |K(x,y)|\nu(dy)/x \in X\} = B$.
- $\|T\|_{p\to p} \leqslant A^{\frac{1}{p}} B^{\frac{1}{p'}}$, $\forall p \in [1,\infty]$ *with* p' *such that* $\frac{1}{p} + \frac{1}{p'} = 1$.
- $\|T\|_{1\to\infty} \leqslant \|K\|_{L^\infty(X\times Y)}$.

Proof :

- Let $f \in L^1(Y,\nu)$.

$$\begin{aligned}
\|Tf\|_{L^1(X,\mu)} &= \int_X |Tf(x)|dx \\
&\leqslant \int_X \int_Y |K(x,y)||f(y)|\nu(dy)\mu(dx) \\
&\underset{\text{F-T}}{=} \int_Y \underbrace{\int_X |K(x,y)|\mu(dx)}_{\leqslant A} |f(y)|\nu(dy) \\
&\leqslant A\|f\|_{L^1(Y,\nu)}.
\end{aligned}$$

So

$$\|T\|_{1\to 1} \leqslant A.$$

- Let $f \in L^\infty(Y,\nu)$.

$$\begin{aligned}
\forall x \in X, |Tf(x)| &\leqslant \int_Y |K(x,y)| \underbrace{|f(y)|}_{\leqslant \|f\|_{L^\infty(Y,\nu)}} \nu(dy) \\
&\leqslant B\|f\|_{L^\infty(Y,\nu)}.
\end{aligned}$$

So

$$\|Tf\|_{L^\infty(X,\mu)} \leqslant B\|f\|_{L^\infty(Y,\nu)}.$$

So

$$\|T\|_{\infty\to\infty} \leqslant B.$$

- $*$ For $p = 1$ or $p = \infty$ we already did it.
  $*$ Let $p \in]1,\infty[$.
  By Riesz-Thorin interpolation (Theorem 2), we have :

$$\|T\|_{p\to p} \leqslant A^{\frac{1}{p}} B^{\frac{1}{p'}}.$$

- Let $f \in L^1(Y,\nu)$.

$$\begin{aligned}
\forall x \in X, |Tf(x)| &\leqslant \int_Y \underbrace{|K(x,y)|}_{\leqslant \|K\|_{L^\infty(X\times Y)}} |f(y)|\nu(dy) \\
&\leqslant \|K\|_{L^\infty(X\times Y)}\|f\|_{L^1(Y,\nu)}.
\end{aligned}$$

So

$$\|T\|_{1\to\infty} \leqslant \|K\|_{L^\infty(X\times Y)}.$$

3.3 The T(1) theorem

3.3.1 The wording

Theorem 8. *Let T be a singular integral operator such that $T, T^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$. Then, the following are equivalent :*

(1) $\begin{cases} T(1_{[0,1]}) \in BMO([0,1]) \\ T(1_{[0,1]}) \in BMO([0,1]), \\ \forall I \in \mathcal{D}^1, \max(\|T(h_I)\|_2, \|T^*(h_I)\|_2) \leqslant \|h_I\|_2 = 1. \end{cases}$

(2) T is bounded on $L^2([0,1])$.

Remark : Once proved this theorem, we can note that we can extend it to any dyadic interval. Hence, $\mathbb{R}$ can replace $[0,1]$ in the theorem.

3.3.2 The proof

• (2) $\Longrightarrow$ (1)

Theorem 9. *Let T be a singular integral operator which is bounded on $L^2([0,1])$.
Then T maps $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$.
Hence $T(1_{[0,1]}) \in BMO([0,1])$ and $T^*(1_{[0,1]}) \in BMO([0,1])$.*

Proof :
Thanks to $BMO - H^1$ duality, it suffices to show that there exists $C > 0$ such that for all atom a, $\|Ta\|_1 + \|T^*a\|_1 \leqslant C$.
Let a be an atom such that $supp(a) \subset I$.
First,

$$\|Ta\|_{L^1(3I)} \underset{\text{Hölder}}{\leqslant} |3I|^{\frac{1}{2}}\|Ta\|_2 \underset{T \text{ bounded on } L^2}{\leqslant} C|3I|^{\frac{1}{2}}\|a\|_2 \underset{\text{atom}}{\leqslant} \sqrt{3}C.$$

Moreover, with y_I the center of I, we have :

$$\begin{aligned}
\int_{\mathbb{R}\setminus 3I} |Ta(x)|dx &\leqslant \int_{\mathbb{R}\setminus 3I}\int_I |K(x,y) - K(x,y_I)||a(y)|dydx \\
&\leqslant C\int_{\mathbb{R}\setminus 3I}\int_I \frac{|I|^\alpha}{dist(x,I)^{1+\alpha}}|a(y)|dydx \\
&\leqslant C\|a\|_1 \int_{\mathbb{R}\setminus 3I} \frac{|I|^\alpha}{dist(x,I)^{1+\alpha}}dx \\
&\leqslant C\|a\|_2|I|^{\frac{1}{2}} \\
&\leqslant C.
\end{aligned}$$

This proves that T maps $H^1([0,1])$ into $L^1([0,1])$.
So by duality T maps $L^\infty([0,1])$ into $BMO([0,1])$, and the theorem falls.

• (1) $\Longrightarrow$ (2)
We consider only the case $T(1_{[0,1]}) = 0 = T^*(1_{[0,1]})$. Actually, the problem can be reduced to this case using paraproducts. The reader can find the solution in [MS113].
We set for all $n \in \mathbb{N}$, $\mathcal{A}_n = \{[(k-1)2^{-n}, k2^{-n}[/1 \leqslant k \leqslant 2^n\}$; $\mathcal{D}_n = \bigcup_{i=0}^{n} \mathcal{A}_i$ (so $\mathcal{D}^1 = \mathcal{D}_\infty$) ; $\Sigma_n = \sigma(\mathcal{A}_n)$; $\mathbb{E}_n(f) = \mathbb{E}[f|\Sigma_n]$ for all $f \in L^1([0,1])$ and $\Delta_n = \mathbb{E}_{n+1} - \mathbb{E}_n$.
By properties of the conditional expectation, we have :

$$\forall n \in \mathbb{N}, \forall f \in L^1([0,1]), \mathbb{E}_n(f) = \sum_{I\in\mathcal{A}_n} 1_I \frac{1}{|I|}\int_I f(x)dx.$$

*

Proposition 7. *$\forall n \in \mathbb{N}, \forall f \in L^1([0,1]), \Delta_n(f) = \sum_{I\in\mathcal{A}_n} \langle f, h_I\rangle h_I$.*

Proof :
We prove it by induction, doing only the case $n = 1$ (because the idea is here) :
For all $f \in L^1([0,1])$,

$$\begin{aligned}
\mathbb{E}_1(f) &= 1_{[0,\frac{1}{2}[}2\int_0^{\frac{1}{2}} f(x)dx + 1_{[\frac{1}{2},1[}2\int_{\frac{1}{2}}^1 f(x)dx \\
&= \int_0^1 f(x)dx + \left(\int_0^{\frac{1}{2}} f(x)dx - \int_{\frac{1}{2}}^1 f(x)dx\right)(1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[}) \\
&= \mathbb{E}_0(f) + \langle f, h_{[0,1[}\rangle h_{[0,1[}.
\end{aligned}$$

$*$ The hypothesis implies that $T\mathbb{E}_0 = \mathbb{E}_0 = 0$ so $Id = \sum_{n=0}^{\infty} \Delta_n$.
Because of the fact that $\mathbb{E}_n = \sum_{m<n} \Delta_m$, we have :

$$T = \underbrace{\left(\sum_{n=0}^{\infty} \Delta_n\right)}_{=Id} T \underbrace{\left(\sum_{m=0}^{\infty} \Delta_m\right)}_{=Id} = \sum_{n=0}^{\infty} (\Delta_n T \Delta_n + \mathbb{E}_n T \Delta_n + \Delta_n T \mathbb{E}_n).$$

Hence, thanks to the triangle inequality, we just have to estimate the quantities $\|\sum_{n=0}^{\infty} \Delta_n T \Delta_n\|_{2\to 2}$, $\|\sum_{n=0}^{\infty} \mathbb{E}_n T \Delta_n\|_{2\to 2}$ and $\|\sum_{n=0}^{\infty} \Delta_n T \mathbb{E}_n\|_{2\to 2}$.
By symmetry, we just have to prove that the first two quantities are bounded. For that we will use the two useful lemmas.
$*$ We focus on the operator $\sum_{n=0}^{\infty} \Delta_n T \Delta_n$.
Thanks to the orthogonality properties of the Haar functions and the previous proposition, we have :

$$\forall (n,m) \in \mathbb{N}^2, n \neq m \Rightarrow (\Delta_n T \Delta_n)(\Delta_m T \Delta_m)^* = 0 = (\Delta_n T \Delta_n)^*(\Delta_m T \Delta_m).$$

As a consequence, we have :

$$\|\sum_{n=0}^{\infty} \Delta_n T \Delta_n\|_{2\to 2} \leqslant \sup_{n\in\mathbb{N}} \|\Delta_n T \Delta_n\|_{2\to 2}.$$

Thus, we have to estimate $\|\Delta_n T \Delta_n\|_{2\to 2}$.
An immediate consequence of definition 13 (as $T \in SIO$) is :

$$\forall I \in \mathcal{D}^1, \forall y \in [0,1]\backslash 3I, |Th_I(y)| + |T^* h_I(y)| \lesssim \frac{|I|^{\frac{1}{2}+\alpha}}{(|I| + dist(y,I))^{1+\alpha}}. \ (\star)$$

For $n \in \mathbb{N}$, let $A^n = (A^n_{I,J})_{(I,J)\in(\mathcal{A}_n)^2}$ be the matrix of the operator $\Delta_n T \Delta_n$ in the basis $(h_I)_{I\in\mathcal{A}_n}$.
We have :

$$\forall (I,J) \in (\mathcal{A}_n)^2, A_{I,J} = \langle h_I, Th_J \rangle.$$

For all $(I,J) \in (\mathcal{A}_n)^2$, we set $k_{I,J} = \min\{k \in \mathbb{N} / \frac{dist(I,J)}{|I|} \leqslant k\}$ (it exists thanks to the fundamental property of $\mathbb{N}$).
Thanks to $(\star)$ and the third hypothesis of the theorem, we have :

$$|A_{I,J}| \lesssim \begin{cases} 1 & \text{if } I = J \\ (1+k_{I,J})^{-\alpha} & \text{otherwise} \end{cases}$$

Hence, Schur's lemma will give that

$$\forall n \in \mathbb{N}, \|\Delta_n T \Delta_n\|_{2\to 2} \leqslant C \text{ where C does not depend on } n.$$

Thus

$$\|\sum_{n=0}^{\infty} \Delta_n T \Delta_n\|_{2\to 2} \lesssim 1.$$

$*$ Now we focus on the operator $\displaystyle\sum_{n=0}^{\infty} \mathbb{E}_n T \Delta_n$.
Let $n \in \mathbb{N}$. Let $B = (B_{I,J})_{(I,J)\in(\mathcal{A}_n)^2}$ be the matrix of the operator $\mathbb{E}_n T\Delta_n$.
Then, for all $(I,J) \in (\mathcal{A}_n)^2$, we have :

$$\begin{aligned}
|B_{I,J}| &= |\langle Th_I, \frac{1}{|J|^{\frac{1}{2}}} 1_J\rangle| \\
&\leqslant |\langle Th_I, \frac{1}{|J|^{\frac{1}{2}}} 1_{3I\cap J}\rangle| + |\langle Th_I, \frac{1}{|J|^{\frac{1}{2}}} 1_{J\backslash 3I}\rangle| \\
&\lesssim 1_{3I\cap J\neq\emptyset} + \int_{J\backslash 3I} \frac{|I|^{\frac{1}{2}+\alpha}}{|J|^{\frac{1}{2}} dist(x,I)^{1+\alpha}} dx \\
&\lesssim 1_{3I\cap J\neq\emptyset} + \frac{|I|^{1+\alpha}}{dist(I,J)^{1+\alpha}} 1_{3I\cap J=\emptyset}.
\end{aligned}$$

Schur's lemma gives us :

$$\sup_{n\in\mathbb{N}} \|\mathbb{E}_n T\Delta_n\|_{2\to 2} \lesssim 1.$$

The difference with what preceeds is that, for all $(n,m) \in \mathbb{N}^2$ such that $n \neq m$, we have

$$(\mathbb{E}_n T\Delta_n)(\mathbb{E}_m T\Delta_m)^* = 0$$

BUT

$$(\mathbb{E}_n T\Delta_n)^*(\mathbb{E}_m T\Delta_m) \neq 0.$$

So, if we want to apply Cotlar's lemma, we have to estimate the last quantity.
Let $(n,m) \in \mathbb{N}^2$ such that $n \neq m$.
We set

$$S_{n,m} = (\mathbb{E}_n T\Delta_n)^*(\mathbb{E}_m T\Delta_m) = (\Delta_n T^*)(\mathbb{E}_{\inf\{n,m\}} T\Delta_m).$$

By symmetry, we can suppose that $m < n$. Hence

$$S_{n,m} = (\Delta_n T^*)(\mathbb{E}_m T\Delta_m).$$

Let $M = (M_{I,J})_{(I,J)\in\mathcal{A}_n\times\mathcal{A}_m}$ be the matix of the operator $S_{n,m}$.

$$\forall (I,J) \in \mathcal{A}_n \times \mathcal{A}_m, M_{I,J} = \langle \mathbb{E}_n Th_I, \mathbb{E}_m Th_J\rangle.$$

For $I \in \mathcal{A}_n$, we set

$$\omega_I(x) = C\left(\frac{|I|}{|I| + dist(x,I)}\right)^{1+\alpha}$$

where C is the constant appearing in $(\star)$.
Then, for all $(I,J) \in \mathcal{A}_n \times \mathcal{A}_m$, we have

$$|\mathbb{E}_n Th_I| \leqslant \frac{1}{|I|^{\frac{1}{2}}}\omega_I \text{ and } |\mathbb{E}_m Th_J| \leqslant \frac{1}{|J|^{\frac{1}{2}}}\omega_J.$$

Thus, thanks to the triangle inequality, we have :

$$\forall (I,J) \in \mathcal{A}_n \times \mathcal{A}_m, |M_{I,J}| \leqslant \frac{1}{(|I||J|)^{\frac{1}{2}}} \int_{\mathbb{R}} \omega_I(x)\omega_J(x)dx.$$

A simple calculus gives us :

$$\forall (I,J) \in \mathcal{A}_n \times \mathcal{A}_m, \int_{\mathbb{R}} \omega_I(x)\omega_J(x)dx \lesssim \min\{|I|,|J|\}\left(\frac{|I|+|J|}{|I|+|J|+dist(I,J)}\right)^{1+\alpha}.$$

So we have :

$$\forall (I,J)\in \mathcal{A}_n\times\mathcal{A}_m, |M_{I,J}|\lesssim 2^{\frac{m-n}{2}}\left(\frac{|J|}{|J|+dist(I,J)}\right)^{1+\alpha}.$$

We set

$$A_{n,m}=\sup_{I\in\mathcal{A}_n}\sum_{J\in\mathcal{A}_m}|M_{I,J}|\lesssim 2^{\frac{m-n}{2}}.$$

The problem is that we can't sum over I. To be able to follow, we have to precise.
We set

$$\mathcal{A}_{n,m}=\{I\in\mathcal{A}_n/\forall J\in\mathcal{A}_m, dist(I,\partial J)\leqslant \lambda|I|\} \text{ and } \mathcal{B}_{n,m}=\mathcal{A}_n\backslash\mathcal{A}_{n,m}.$$

The constant λ has to be chosen with care.
We set

$$B^1_{n,m}=\sup_{J\in\mathcal{A}_m}\sum_{I\in\mathcal{A}_{n,m}}|M_{I,J}|\lesssim 2^{\frac{m-n}{2}}\lambda.$$

Now we have to estimate the sommation over $\mathcal{B}_{n,m}$.
We have :

$$\int_0^1\mathbb{E}_nTh_I(x)dx=\int_0^1 Th_I(x)dx=\langle 1_{[0,1]},Th_I\rangle=\langle \underbrace{T^*1_{[0,1]}}_{=0},h_I\rangle=0.$$

Let $I\in\mathcal{A}_n$. Let $I_a\in\mathcal{A}_m$ be the unique ancestor of I (it exists because $m<n$).
For all $j\in\mathcal{A}_m$, we set :

$$(Th_J)(I_a)=\frac{1}{|I_a|}\int_{I_a}\mathbb{E}_mTh_J(x)dx=\frac{1}{|I_a|}\int_{I_a}Th_J(x)dx.$$

Then, for all $(I,J)\in\mathcal{B}_{n,m}\times\mathcal{A}_m$, we have :

$$\begin{aligned}
|M_{I,J}| &= |\int_{I_a}(\mathbb{E}_nTh_I(x))(Th_J)(I_a)dx+\int_{I_a^c}(\mathbb{E}_nTh_I(x)))(\mathbb{E}_mTh_J(x))dx| && (42)\\
&= |-\int_{I_a^c}(\mathbb{E}_nTh_I(x))(Th_J)(I_a)dx+\int_{I_a^c}(\mathbb{E}_nTh_I(x))(\mathbb{E}_mTh_J(x))dx| && (43)\\
&\leqslant C\int_{I_a^c}\frac{1}{|I|^{\frac12}}\omega_I(x)dx\frac{1}{|I_a|}\int_{I_a}\frac{1}{|J|^{\frac12}}\omega_J(x)dx+\int_{I_a^c}\frac{1}{(|I||J|)^{\frac12}}\omega_I(x)\omega_J(x)dx && (44)\\
&\lesssim \left(\frac{|I|}{|J|}\right)^{\frac12}\left(\frac{|J|}{|J|+dist(I_a,J)}\right)^{1+\alpha}\frac{1}{\lambda^\alpha}. && (45)
\end{aligned}$$

Thus we can set

$$B^2_{n,m}\sup_{J\in\mathcal{A}_m}\sum_{I\in\mathcal{B}_{n,m}}|M_{I,J}|\lesssim 2^{\frac{n-m}{2}}\frac{1}{\lambda^\alpha}.$$

Hence

$$A_{n,m}(B^1_{n,m}+B^2_{n,m})\lesssim 2^{m-n}\lambda+\frac{1}{\lambda^\alpha}.$$

For a good λ, we have finally :

$$A_{n,m}(B^1_{n,m}+B^2_{n,m})\lesssim 2^{\frac{(n-m)\alpha}{1+\alpha}}.$$

We just have to apply Cotlar's lemma to end the proof of this theorem.

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